Heat kernel smoothing on manifolds and its application to longitudinal brain substructure modeling

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Abstract

We present a novel kernel smoothing framework on an arbitrary manifold using the Laplace-Beltrami eigenfunctions. The Green’s function of an isotropic diffusion equation on a manifold is analytically represented first using the eigenfunctions of the Laplace-Beltrami operator. The Green’s function is then used in constructing heat kernel smoothing analytically. Unlike many previous diffusion based smoothing approaches developed for manifolds data, diffusion is analytically solved avoiding various numerical instability and inaccuracy issues. Our method is compared to a widely used iterative kernel smoothing technique in brain imaging to show significant improvement in numerical accuracy. The proposed framework is illustrated with longitudinally collected hippocampus surfaces.
Effect of family income on hippocampus growth

86 teens from high income family (> $75000)
mean age = 12 +/- 4 years

38 teens from low income family (< $35000)
mean age= 12 +/- 4 years old

Each subject has multiple scans (1-2 scans).
Image Processing
Image registration
Deformation from the template to a subject.
Deformation form the template to Scan2 is given by warp1 + warp2.
Manual hippocampus segmentation on MRI template
Subcortical structures: hippocampus
surface template & deformation on template

Deformation field of warping the template to a subject
Heat kernel smoothing on manifolds

Original idea given in Chung et al. 2005 (NeuroImage).

The most widely used cortical data smoothing technique in brain imaging.

Similar iterative smoothing methods are now implemented in FreeSurfer, AFNI, SurfStat.
Approximating heat kernel with Gaussian kernel

Parametrix expansion (Rosenberg, 1997):

\[
K_\sigma(p, q) = \frac{1}{(4\pi\sigma)^{1/2}} e^{-\frac{d^2(p, q)}{4\sigma}} \left[ 1 + O(\sigma^2) \right]
\]

for small bandwidth

\[
K_{k\sigma} * f = K_\sigma * \cdots * K_\sigma * f
\]

\(k\) times
Validation against the ground truth

Chung et al., 2008. Statistica Sinica

Can we do better?
Laplace-Beltrami eigenfunctions
Eigenfunctions of Laplace-Beltrami operator

Helmholtz equation

$$\Delta \psi_j = \lambda_j \psi_j$$
Generalized eigenvalue problem via FEM

\[
\Delta f = \lambda f \quad \rightarrow \quad C \psi = \lambda A \psi
\]

Qiu et al. IEEE TMI 2006
Seo et al. MICCAI 2010
First 10 orthonormal basis on left hippocampus

MATLAB code available from my website.
Laplace-Beltrami eigenfunction expansion

\[ x_i(p) = \sum_{j=0}^{k} f_{ij} \psi_j(p) \]
Reconstruction error

Cortical surfaces
Amygdala surfaces

Red= LB-expansion
Blue= SPHARM

Seo et al. 2010. ISIB
Heat kernel smoothing on manifold

Heat kernel:

\[ K_t(p, q) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \psi_i(p) \psi_i(q) \]

Heat kernel:

\[ K_t \ast f = \int_{\mathcal{M}} K_t(p, q) f(q) \, dq \]

x-coordinate on mandible surface

smoothed with bandwidth 10 and 1269 eigenfunctions
Heat kernel smoothing on hippocampus surface

Original deformation field

Heat kernel smoothing with bandwidth = 1
Mixed Effect Modeling

$y_{ij}$  $i$-th subject, $j$-th scan ($j=1,2$)

**Fixed effect model:**

$$y_{ij} = \beta_0 + \beta_1 \text{age}_{ij} + \epsilon_{ij}$$

**Mixed effect model:**

$$y_{ij} = \beta_0 + \gamma_{i0} + (\beta_1 + \gamma_{i1}) \text{age}_{ij} + \epsilon_{ij}$$

Each subject has its own growth intercept and slope.
Results
Effect of family income on hippocampus growth

Chung et al. HBM 2011

p-value = 0.05

p-value = 0.025
Conclusion

Low income level
→ Adverse environment
→ Stress
→ Abnormal hippocampus growth
Next Project
Second eigenfunction of elongated objects

*Hot spot conjecture in differential geometry:*
Min and max of the second eigenfunction always occur at the extreme end points or boundary
Rauch’s hot spot conjecture unsolved since 1974:

\[ K_\sigma \ast f(p) = \int_\mathcal{M} \frac{f(p) \, d\mu(p)}{\mu(\mathcal{M})} + f_1 e^{-\lambda_1 \sigma} \psi_1(p) + R(\sigma, p) \]

The heat kernel smoothing asymptotically behaves like the second eigenfunction.

Mean signal over manifold

Error term
Thank you

MATLAB codes can be downloaded from www.stat.wisc.edu/~mchung