Applications of Fourier Transform to Imaging Analysis

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Abstract

In this report, we propose a novel automatic and computationally efficient method of Fourier imaging analysis using Fourier transform. Besides Fourier transform's many applications, one can use Fourier transform to select significant frequencies of an observed noisy signal, which can be applied as a model selection tools of (weighted) Fourier series analysis of medical images. Both simulated data and Corpus Callosum (CC) data are used to demonstrate the advantages of our method over previous methods. The possibilities of applications of this method to image analysis is discussed.

1 Introduction

Fourier transform (FT) is named in the honor of Joseph Fourier (1768-1830), one of greatest names in the history of mathematics and physics. Mathematically speaking, The Fourier transform is a linear operator that maps a functional space to another functions space and decomposes a function into another function of its frequency components. The formulae used to defined Fourier transform vary according to different authors (Arfken, 1985, Krantz, 1999 and Trott, 2004). But they are essentially the same but using different scales. In this report, we are using the definition in Bracewell, 1999, which is widely used in many literatures (e.g. Brigham, E.O., 1988, Körner, 1988, Sogge, 1993 and Kammler, 2000) . Suppose $g \in L(\mathbb{C}), \mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$. *Fourier transform* is a linear operator $F : L(\mathbb{C}) \to L(\mathbb{C})$ defined as

$$G(w) = Fg(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{-iwt}dt, \quad w \in \mathbb{R}.$$

If g is sufficiently smooth, then it can be reconstructed from its Fourier transform using the *inverse Fourier transform*

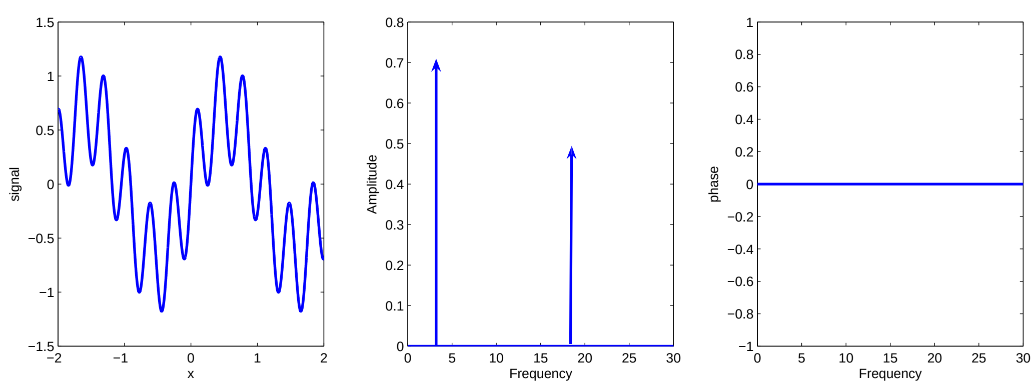

Figure 1. The amplitude (left) and phase function of the Fourier transform of $g = 0.7\sin(3x) + 0.5\sin(18x)$.

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$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(w) e^{iwt} dw.$$

The existence of inverse Fourier transform tells us that, for certain conditions, a function can be uniquely represented by its Fourier transform. For the purpose of interpretation and visualization, Fourier transform $G(w)$ is usually expressed in polar coordinate as $G(w) = A(w) \cdot e^{ip(w)}$, where we call $A(w) = \|G(w)\|$ the amplitude function and $p(w) = \angle G(w)$ the phase function (as shown in Figure 1).

Fourier transform, which was first proposed to solve PDEs such as Laplace, Heat and Wave equations, has enormous applications in physics, engineering and chemistry. Some applications of Fourier transform include (Bracewell, 1999)

1. *communication:* Fourier transform is essential to understand how a signal behaves when it passes through filters, amplifiers and communications channels (Chowning, 1973, Brandenberg and Bosi, 1997 and Bosi and Goldberg, 2003).
2. *image processing:* Transformation, representation, and encoding, smoothing and sharpening images.
3. *data analysis:* Fourier transform can be used as high-pass, low-pass, and band-pass filters and it can also be applied to signal and noise estimation by encoding the time series (Good, 1958, 1960, Harris, 1978, Zwicker and Fastl, 1999, Kailath, *et al.*, 2000 and Gray and Davisson, 2003).

In this report, we focus on the applications of Fourier transform to image analysis, though the techniques of applying Fourier transform in communication and data process are very similar to those to Fourier image analysis, therefore many ideas can be borrowed (Zwicker and Fastl, 1999, Kailath, *et al.*, 2000 and Gray and Davisson, 2003). Similar to Fourier data or signal analysis, the Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components. Comparing with the signal process, which is often using 1-dimensional Fourier transform, in imaging analysis, 2 or higher dimensional Fourier transform are being used. Fourier transform has been widely applied to the fields of image analysis.

Image segmentation is one of the most widely studied problem in image analysis. Many literatures also found that Fourier transform can be used effectively in image segmentation. Shu *et al.* (1992) presented an efficient algorithm to compute the critical dimensions of aligned rectangular and trapezoidal wafer structures using images generated by a Fourier imaging system. Paquet *et al.* (1993) introduced a new approach for the segmentation of planes and quadrics of a 3-D range image using Fourier transform of the phase image. Li and Wilson (1995) established a Multiresolution Fourier Transform to approach the segmentation of images based on the analysis of local information in the spatial frequency domain. Wu *et al.* (1996) presented an iterative cell image segmentation algorithm using short-time Fourier transform magnitude vectors as class features. Escofet *et al.* (2001) applied Fourier transform to image segmentation and pattern recognition. Zou and Wang (2001) proposed a method to exploit the auto-registration property of the magnitude spectra for texture identification and image segmentation. Our method can potentially be applied to many of those previous segmentation problems.

Fourier transform image classification techniques were also widely used. Robert (1980) introduced The discrete Fourier transform (DFT) automated satellite imagery classification technique is designed to detect and identify cloud features from 25 x 25 nautical mile (nm) Defense Meteorological Satellite Program (DMSP) visible and infrared imagery samples. Levchenko *et al.*, 1992 designed a neural network for image Fourier transform classification. Harte and Hanka, 1997, designed an algorithm for large classification problem using *Fast Fourier Transform* (FFT). This paper was trying to deal with curse of dimensionality problem, which is the purpose of this paper too. Tang and Stewart, 2000 used Fourier transform to classify optical and sonar images. The classification performance of Fourier transform was compared with that of wavelet packet transform. Kunttu *et al.* (2003) applied Fourier transform to perform image classification.

Lustig et al. (2004) presented a fast and accurate discrete spiral Fourier transform and its inverse. The inverse solves the problem of reconstructing an image from MRI data acquired along a spiral k-space trajectory. Rowe and Logan (2004), Rowe (2005) and Rowe *et al.* (2007) used Fourier transform to reconstruct signal and noise of fMRI data utilizing the information of phase functions of Fourier transform of images.

Most papers described above have one thing in common: they did not talked about how to choose the important frequencies of Fourier transform. While some of them discussed or focused on how to choose the frequencies up to certain degrees and used those frequencies to represent the signals. Mezrich (1995) propose an imaging modalities that one can choose the dimension of K-space and therefore choose the proper number of frequencies of the observed signal. Wu *et al.* (1996) obtained the K-space using so called "short-time Fourier transform magnitude vectors". Lustig et al. (2004) also proposed a fast spiral Fourier transform to effectively choose the K-space. Li and Wilson (1995) proposed Laplacian pyramid method to filter out the high frequencies by using a unimodal Gaussian-like kernel to convolve with images. The problem with those selection methods and procedures did not work on the possibility that even some low frequencies are not necessarily important. And after picked up the important frequencies, they chose inverse Fourier transform to reconstruct the signal. While for some cases, using the Fourier transform itself, we can construct the signal by applying the Fourier transform to Fourier series analysis.

In this report, we are going to propose a method that using Fourier transform as model selection tool to do Fourier image (in Section III) based on the important properties of Fourier transform (in Section II). And some uncomplete works, possible works and how we may apply our method to various image analysis procedures are presented in the Discussions (Section IV).

2 Properties of Fourier Transform

The applications of Fourier transform are abased on the following properties of Fourier transform.

Theorem 2.1 *For a given abounded continuous integrable function (e.g. f), we denote the corresponding capitol letter (e.g. F) as its Fourier transform.*

a. *if* $g(x) = f(x-a)$, *then* $G(w) = e^{-iaw}F(w)$.

b. *If* $g(x) = f(x/\lambda)$, *then* $G(w) = \lambda F(\lambda w)$.

c. *If* $h = f * g$, *then* $H(w) = F(w)G(w)$.

d. *If* $d(x) = f'(x)$, *then* $D(w) = iwF(w)$.

e. *If* $f(x) = \cos(2\pi w_0 x)$, *then* $F(w) = \delta(w + w_0) + \delta(w - w_0)$; *If* $f(x) = \sin(2\pi w_0 x)$, *then* $F(w) = \delta(w + w_0) + \delta(w - w_0)$.

The above properties can be used to find the solution of heat equation with initial values as stated in the following theorem.

Theorem 2.2 *Let f_0 be a bounded integrable function in $\mathbb{R}^n$. The unique solution to the heat equation*

$$\begin{cases} f_t - \Delta f &= 0, \quad t > 0 \text{ and } x \in \mathbb{R}^n \\ f(x,0) &= f_0 \end{cases}$$

is given by $f(x,t) = h * f_0$, *where* $h = e^{-\|x\|^2/t}$.

Proof. By Fourier expansion, $f(p) = \sum_{j=0}^{\infty} \langle f, \phi_j \rangle \phi_j(p)$. Fourier transform yields $G_t(w,t) = \|w\|^2 G(w,t)$. Then $G(w,t) = e^{-t\|w\|^2}F(w)$. Note that the Fourier transform of $e^{-\|x\|^2/t}$ is $e^{-t\|w\|^2}$ and properties c in Theorem 2.1, one ten use inverse Fourier transform to finish the proof. □

3 One-dimensional Fourier analysis using Fourier Transform

In this report, we are going to apply these properties to Fourier analysis of image analysis. Let $\mathcal{M}$ be a compact manifold. Hilbert space $L^2(\mathcal{M})$ is defined with an inner product

$$\langle f, g\rangle = \frac{1}{\mu(\mathcal{M})}\int_{\mathcal{M}} f(x)g(x)d\mu(x).$$

Let $\{\phi_j\}_{j=1}^{\infty}$, a complete orthnormal basis of $L^2(\mathcal{M})$. Then the Fourier series of $f \in L^2(\mathcal{M})$ of a given signal f is

$$f(p) = \sum_{j=1}^{\infty}\langle f, \phi_j\rangle\phi_j(p), \quad p \in \mathcal{M}.$$

A most widely used special case of Fourier series is Fourier expansion. Let $\mathcal{M}$ be the unit circle. A complete orthonormal basis of $L^2(\mathcal{M})$ is given as $\{1/2, \cos(ix), \sin(ix), i = 1, 2, \cdots\}$. Then any function $f \in L^2(\mathcal{M})$ can have the Fourier expansion as

$$g = \frac{a_0}{2} + \sum_{i=1}^{\infty}(a_i\cos(ix) + b_i\sin(ix))$$

where

$$\begin{aligned} a_i &= \frac{1}{\pi}\int_0^{2*\pi} f\cos(ix)dx \\ b_i &= \frac{1}{\pi}\int_0^{2*\pi} f\sin(ix)dx. \end{aligned}$$

Using Theorem 2.1, for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$ we have

$$\begin{aligned} F(w) &= \frac{a_0}{2}\delta(w) + \sum_{n=1}^{\infty}(a_n(\delta(w+n) - \delta(w-n)) \\ &\quad +b_n(\delta(w+n) - \delta(w-n))) \\ &= \frac{a_0}{2}\delta(w) + \sum_{n=1}^{\infty}((a_n+b_n)\delta(w+n) + (a_n-b_n)\delta(w-n)) \end{aligned} \tag{1}$$

So from the amplitude of Fourier transform of f, we can figure out all the $\{a_0, a_1, b_1, \cdots\}$.

But traditionally, (Chung *et al.*, 2007), to estimate $\boldsymbol{f} = \sum_{i=1}^{K}\exp(-\lambda_i t)\beta_i\phi_i + \boldsymbol{\epsilon}$, we first estimate $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_K)$ in model

$$\boldsymbol{f} = \boldsymbol{Y}\boldsymbol{\beta} + \boldsymbol{\epsilon}. \tag{2}$$

where $\boldsymbol{f}$ is the interested curve or surface, and the design matrix

$$\boldsymbol{Y} = \begin{bmatrix} \phi_1(p_1) & \cdots & \phi_K(p_1) \\ \vdots & \ddots & \vdots \\ \phi_1(p_n) & \cdots & \phi_K(p_n) \end{bmatrix}$$

$\{\phi_i\}_{i=1}^{k}$ are Fourier or spherical harmonic basis functions. The least squares estimation (LSE) of $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{Y}'\boldsymbol{Y})^{-1}\boldsymbol{Y}'\boldsymbol{f}. \tag{3}$$

One problem with LSE is that LSE is very computational inefficient, especially for those high frequency signals as we are going to show in our first simulation. For medical images, the data can be extremely large, which means the design matrices can be too large for computers' memory to load and it is impossible to operate these large matrices. For LSE method, one also has to calculate the inverse matrices. General PCs can not handle the computation of the inverse of very large matrices. To handle this problem, *Iterative Residual Fitting* (IRF) (Chung *et. al*, 2007), are proposed. But the limitations of IRF are no model selection procedure and no stopping rules.

The problem of Fourier analysis of one-dimensional signals can be modeled as follows: suppose one trying to estimate signal $g(x), x \in [0, 2\pi]$. Only noisy signal is observed

$$g_1(x) = g(x) + \epsilon(x)$$

where $\epsilon(x) \sim N(0, \sigma^2)$ is the white noise; one is trying to find the Fourier series representation of the true signal

$$\hat{g} = \frac{a_0}{2} + \sum_{n=1}^{K} (a_n \cos(nx) + b_n \sin(nx))$$

where K is selected manually or automatically.

In this report, we propose an alternative method of Fourier analysis using Fourier transform. Firstly, we calculate the Fourier transform of an observed signal (or function). Then, using the following selection rule to select the most important frequencies: a frequency is selected if

$$\|\frac{\sqrt{n}(F(w) - \text{Mean } F(w))}{\text{Std } F(w)}\| > t_{n-1,0.005}$$

The intuition of this method is that if we suppose $F(w)$ are normally distributed, we pick the frequencies that contribute to the Fourier transform "Significantly" at 0.01 level, which means we miss an important frequency with a probability less than 0.01 . After we pick up the significant frequencies, we finally use (1) to approximate the true signal. Our following simulations will also give clear demonstration of our procedure.

Our first simulation is to estimate the sinusoid signal. For this simulation, we let

$$g_1(x) = 0.7 \sin(7x) + \sin(18x) + \epsilon$$

where $\epsilon \sim N(0, 0.2^2)$ as shown in Figure 2.

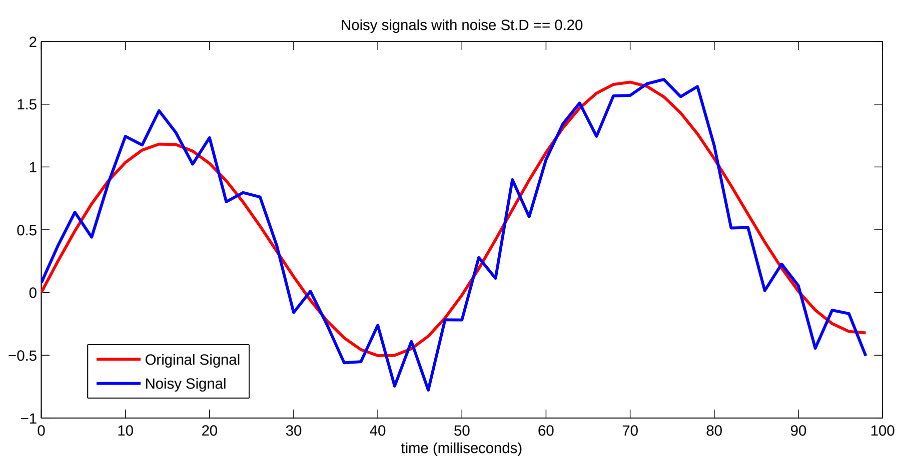

Figure 2. Original and noisy curve used in the simulation.

So as we see, to estimate the signal $g(x)$ using Least-squares estimation, one has to generate at least $2 \times 18 + 1$ basis function. On the other hand, using Fourier transform, one can easily find that 2 basis functions are enough for our analysis. And at the same time, Fourier transform gives the estimation of coefficients of the corresponding basis functions as shown in Figure 6. So we see that, when we using 1000 observations, the amplitudes are not exactly at 0.7 and 1. The main reason is because of the presence of noise. The other reason is that one has finite range of observations while the Fourier transform is defined over the whole real line. If one increase the range of observations, as shown in Figure 3, we can have much better FT results.

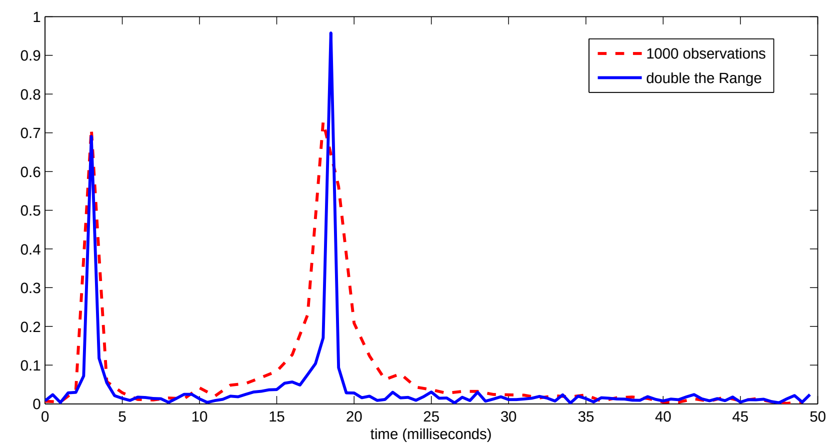

Figure 3. Fourier transform results using diffrent observation range.

The using the results of Fourier transform, we estimate the signal function as shown in Figure 4. We see that, when using 1000 observations, the estimation is over-smoothing. But if we increase the range of observations, we have a very good estimation of the original signal.

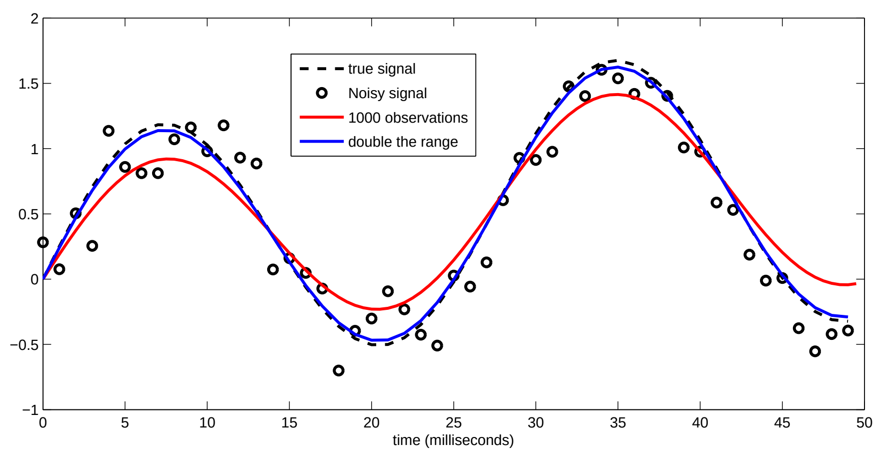

Figure 4. Curve estimation using Fourier transform.

From above simulation, we see that, for estimation of trigonometric functions or their combinations, Fourier transform will give really good and really fast results comparing with Least-squares estimation. Consider that if the signal function has one part as $\sin(nx)$ when n is very large, the Least-squares estimation will be very inefficient to use all the $2n + 1$ basis functions.

Now, let's check how the Fourier transform works on more general signal estimation. Let true signal be

$$g(x) = x^2 \cdot (x - 2\pi)^2, \quad x \in [0, 2\pi].$$

Note that $g(x)$ is periodic and smooth (its first derivative is continuous) as shown in Figure 5. And for the general curve that we defined above, one still find the fitting is very good and it is very fast as shown in Figure 6.

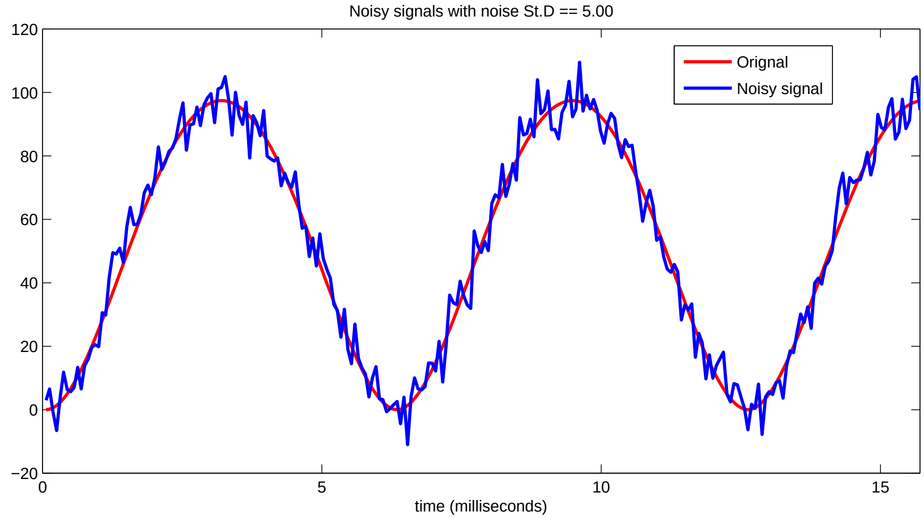

Figure 5. A non-trigonometric curve

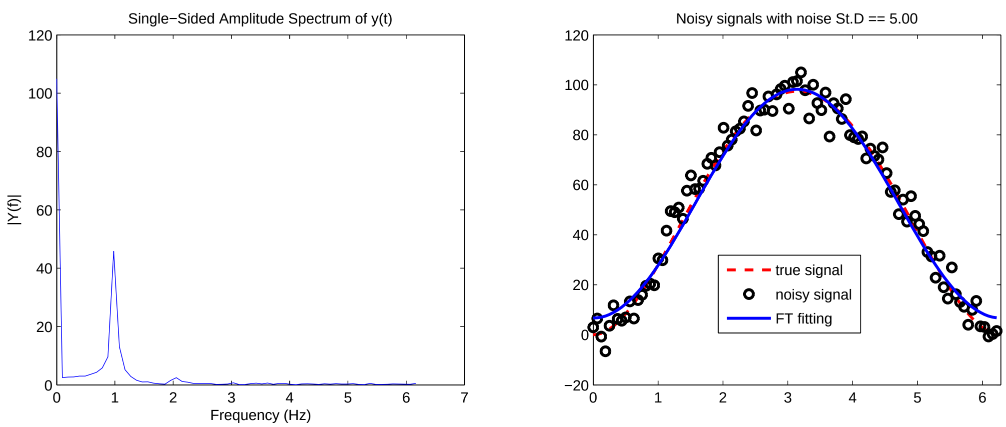

Figure 6. The FT estimation of non-trigonometric curve

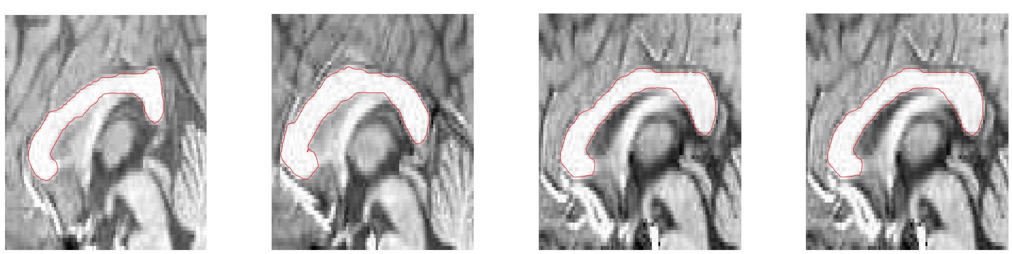

Figure 7. Some results of GVF snakes.

We finally apply our method to the CC data. The boundaries of CC's are extracted using Gradient Vector Flow snakes (Xu and Prince, 1998) as shown in Figure 7. As one can see, that GVF snakes can converge to the concave parts of the boundaries, thus capture the detailed information of the boundaries. At the meantime, GVF snakes will provide noisy boundaries of CC's. So a smooth CC boundaries should be provided. Firstly, using arc-length parametrization, for each obtained discrete curve $\{p_i\}_{i=1}^n$, we have

$$C(s_i) = (x(s_i), y(s_i)), \quad 0 = s_1 < s_2 \cdots < s_n = 2\pi.$$

Then, we are going to perform Fourier analysis on two curve $x(s)$ and $y(s)$, $s \in [0, 2\pi]$ as shown in Figure 8. And Figure 9, 10 show the results of Fourier analysis using Fourier transform. We see that FT gave comparable results to LSEs, while FT used fewer basis functions.

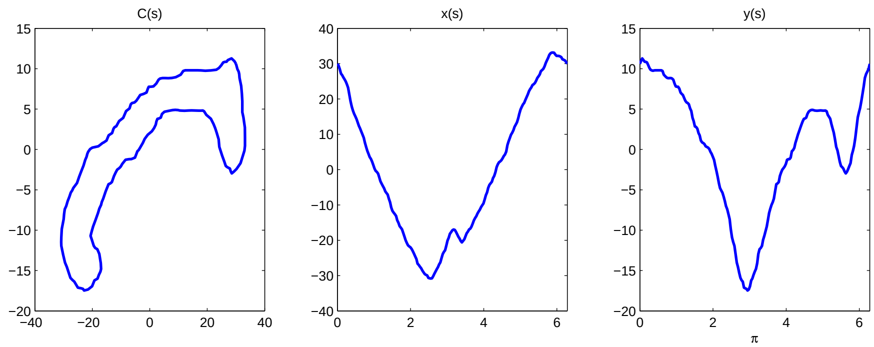

Figure 8. The closed curve decomposed into two functions.

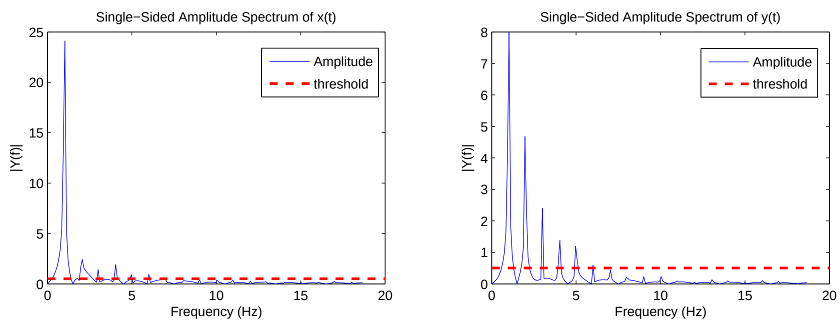

Figure 9. The amplitude functions of $x(s)$ and $y(s)$.

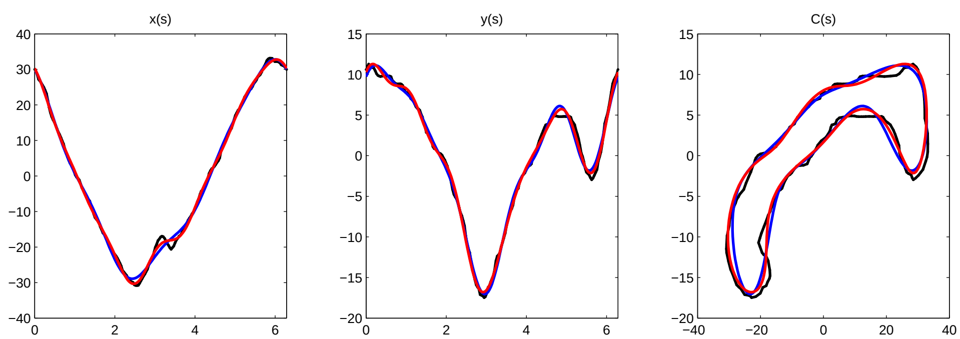

Figure 10. The reconstructed boundary of CC. The black curves are the observation, the blue curves are FT results and the LSEs of degree 6 Fourier expansion are red curves.

4 Discussions

From the Fourier analysis of simulated data and CC boundaries using Fourier transform, we conclude that our method has the following advantages over LSE:

1. *Automatics model selection:* we gave an automatic model selection procedure that is similar to the classical outlier tests in Statistics. Its validation still need to be investigate thoroughly later.

2. *Computation efficiency:* By the model selection procedure, our method always pick only significant frequencies, which can save us a lot computation. For some cases (such as the one in our first simulation), our method can show extreme advantage over LSE. And to numerically computation to Fourier transform, we are using Fast Fourier Transform, which is first introduced by Cooley and Tukey, 1965, the Cooley-Tukey algorithms. Then this algorithm was widely accepted, implemented and improved (Nassbauner, 1982, Ramirez, 1985, Gentleman and Sande, 1996 and Walker, 1996). The bound on complexity and operation counts is $O(N\log(N))$, where N is the number of observations. For LSE, the bound on complexity and operation counts is $O(P^3)$, where P is the

number of basis. So these two methods are not comparable the bounds of complexity and operation counts in all cases. But since the order of LSE is 3, which is much higher $N\log(N) = o(N^2)$, so at least for some cases, our method should be faster than LSE. Harte and Hanka also used Fourier transform to deal with large scaled problem. The performance of their method and ours can be compared later.

We can also generalize our method to two-dimensional Fourier analysis using Fourier transform. Suppose we have closed surfaces

$$S(\theta,\phi) = (X(\theta,\phi), Y(\theta,\phi), Z(\theta,\phi)).$$

So Fourier transform is performed on $X(\theta,\phi), Y(\theta,\phi), Z(\theta,\phi)$ individually.

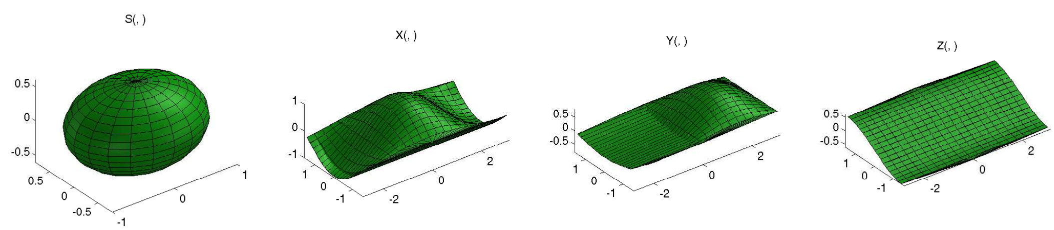

Figure 10. The closed surface decompose into three surfaces.

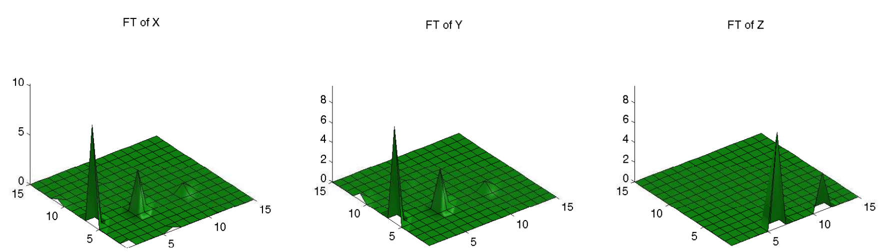

Figure 11. The amplitude functions of $X(\theta,\phi)$, $Y(\theta,\phi)$ and $Z(\theta,\phi)$.

Therefore, we can perform the similar model selection procedure to select importance "pikes" as shown in Figure 11. But at this moment, the reconstruction formulae are not very clear yet. One still need to find the Fourier transform of similar format to (1).

If one can specify the format of Fourier transform on the unit sphere S^n, one can also find another proof (similar to the proof of Theorem 2.1) of the following important theorem in Weighted Fourier Analysis, which is a generalized version of Theorem 2.1.

Theorem 4.1 *(Moo et al., 2007) Consider a Cauchy problem of linear operator $\mathcal{L} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$,*

$$\begin{cases} \frac{\partial g(p,t)}{\partial t} + \mathcal{L}g(p,t) &= 0, \qquad t \geq 0 \\ g(p,0) &= f(p). \end{cases}$$

It defines a natural smoothing procedure. t controls the amount of smoothing and is termed as the bandwidth. The unique solution to (1) is given as

$$g(p,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\langle f, \phi_j\rangle \phi_j(p),$$

where λ_j's and ϕ_j's are the eigenvalues and eigenfunctions of $\mathcal{L}$. We call $g(p,t)$ is a Weighted Fourier Series (WFS).

Theorem 4.1 can be proved using the Fourier series of f_0 and the plug into the equation too.

Therefore, one may apply our method to image segmentation as a tool to select basis functions for the variational problem (like snake algorithm using splines). While, unlike what people before, the number of basis function are more flexible and thus the computation is more efficient based on our method. Using the magnitudes of Fourier transform, one may represent the functional signals as multivariate signals and thus able to apply linear classification methods. And our method itself is already a fast and efficient way to reconstruct signals or images. While all those have to be tested thoroughly in more rigorous ways in the future research.

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