Computational Methods in NeuroImage Analysis

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Lecture 12

Complexity in network and brain images

November 26, 2010

Class schedule:

Take home exam on December 2. Due to popular demand, exam schedule has been changed.
There is no class on December 3. Exam will be emailed to you by December 3 9:00am or earlier. Your solution should be emailed to me in PDF with accompanying MATLAB code (zip it) by December 6 9:00am. It will take minimum 30 hours to finish.

Oral exam on December 10.
Send me your talk title by next December 2.

Final report due December 17.
Send PDF by 9:00pm. There will be penalty for late reports. Do not submit twice. First submission will be graded.

Image Complexity

- Motivation: why we are interested in studying image complexity?
- Image complexity can characterize an image and the underlying clinical status (existence of cancer, Alzheimer's disease).
- The most widely used image complexity measure is the ***fractal dimension (FD).***
- We will study ***fractals*** and ***fractal dimension***.

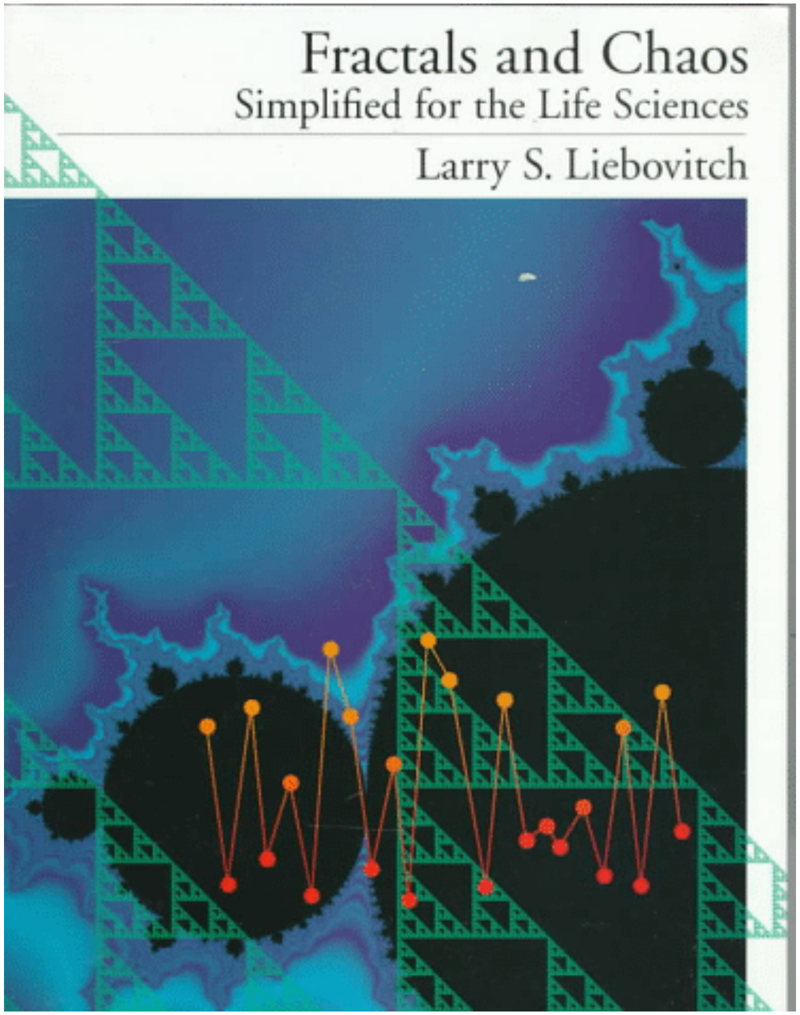
Use of Fractals

- Quantification of biological systems (most complex and chaotic in science).

- Fractal image compression: it can achieve the compression ratio of up to 600:1.

- Fractal rendering: realistic computer rendering of clouds, rocks, shadows

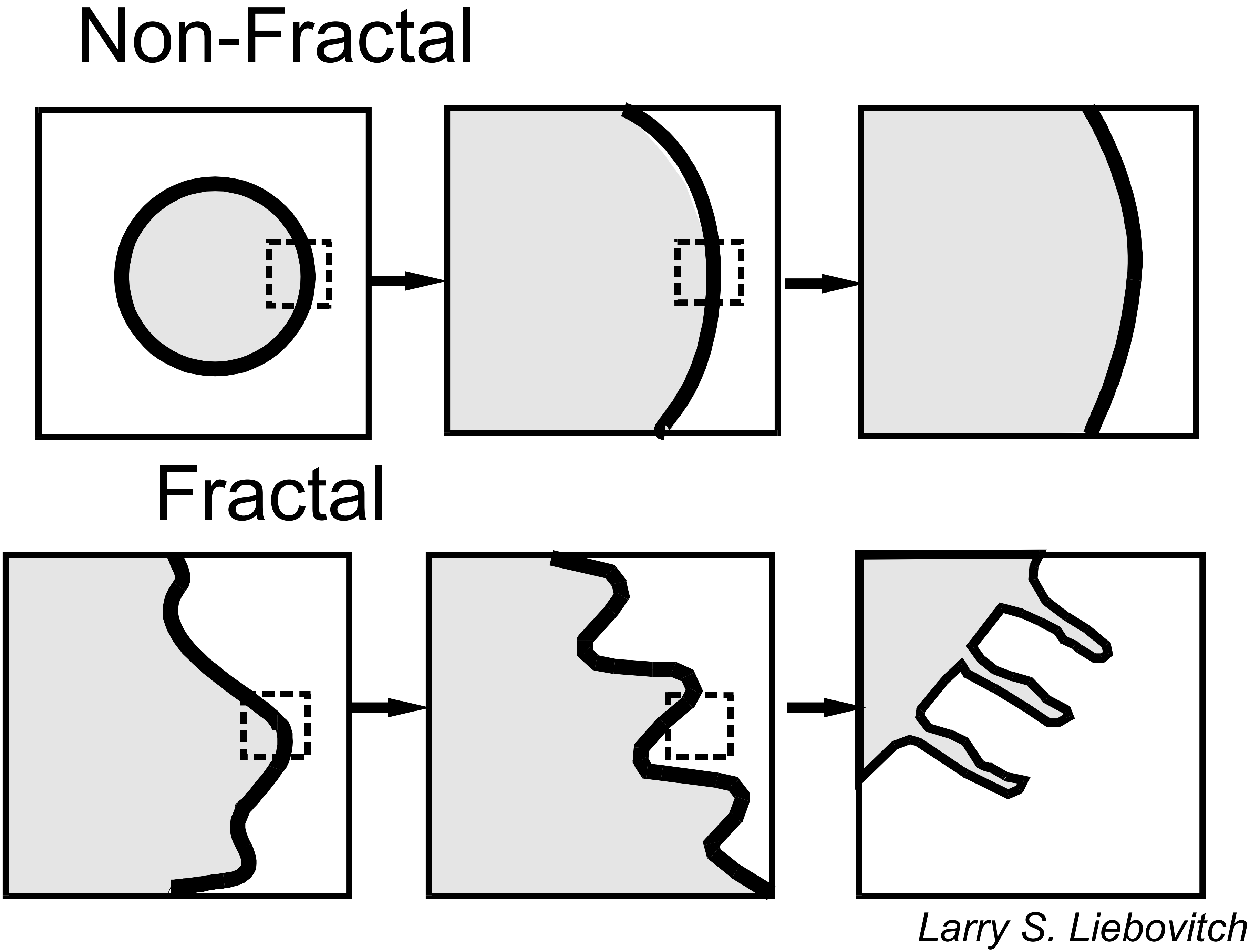
References:



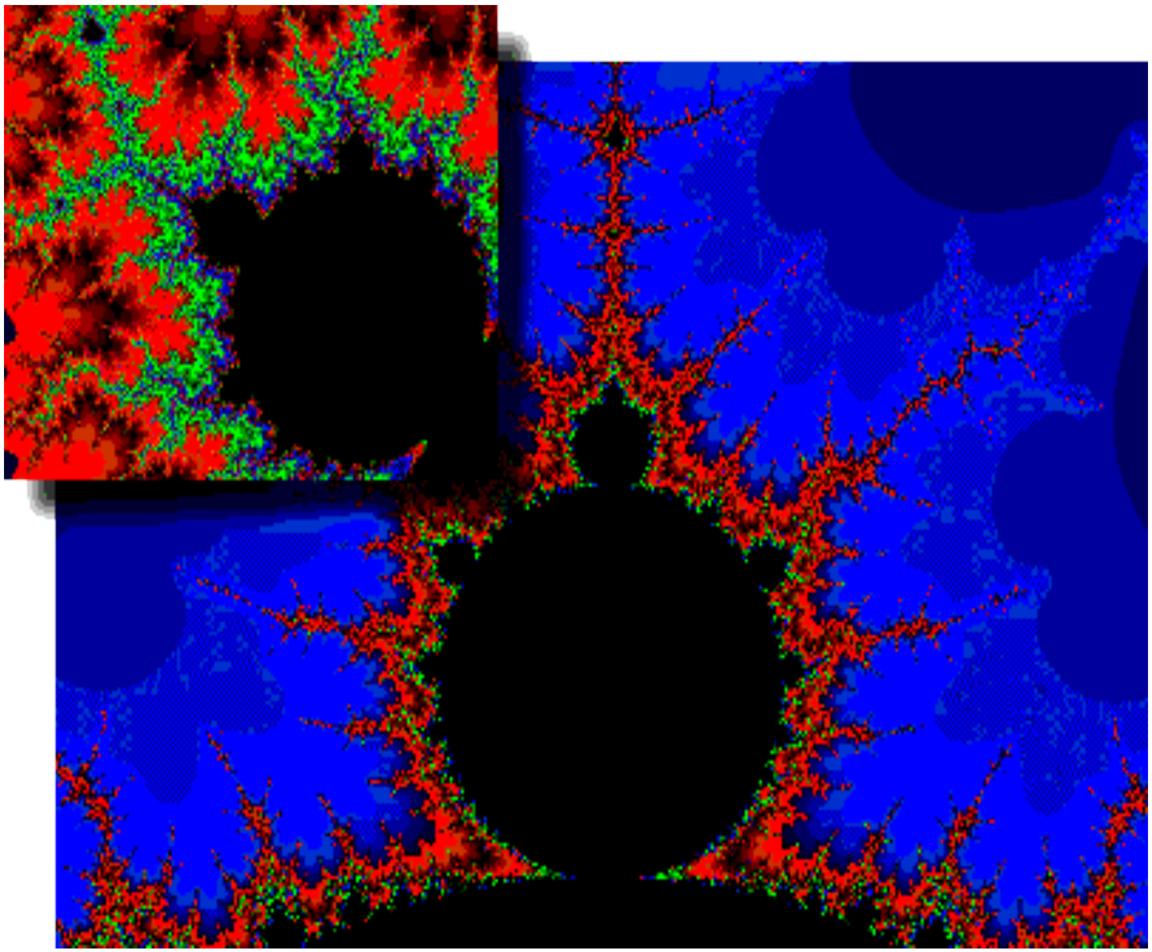
Fractals and Chaos and
Simplified for the Life Sciences

Larry S. Liebovitch
Oxford Univ. Press, 1998

www.ccs.fau.edu/~liebovitch/larry.html


Non-Fractal
Fractal
Larry S. Liebovitch

Examples of fractals



Fractals can be generated by recursive formula in a complex plane.

For instance, the following formula will generate the left fractal:

$$Z_{n+1} = Z_n^{\ 2} + C$$

History of fractals

- Benoit B. Mandelbrot (1960). *Fractal geometry of nature*

- Mandelbrot termed ***fractals*** for geometric objects with self similarity.

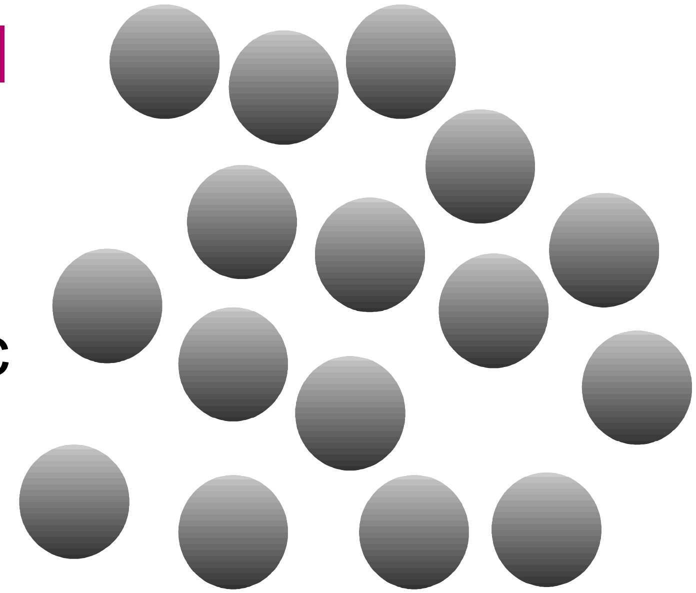
Properties of Fractals

- Infinite detail at every point of the object
- Self (affine) similarity between parts and overall features of the object.
- Zoom into traditional shape: less detail
- Zoom into fractals: more detail

Size of Features

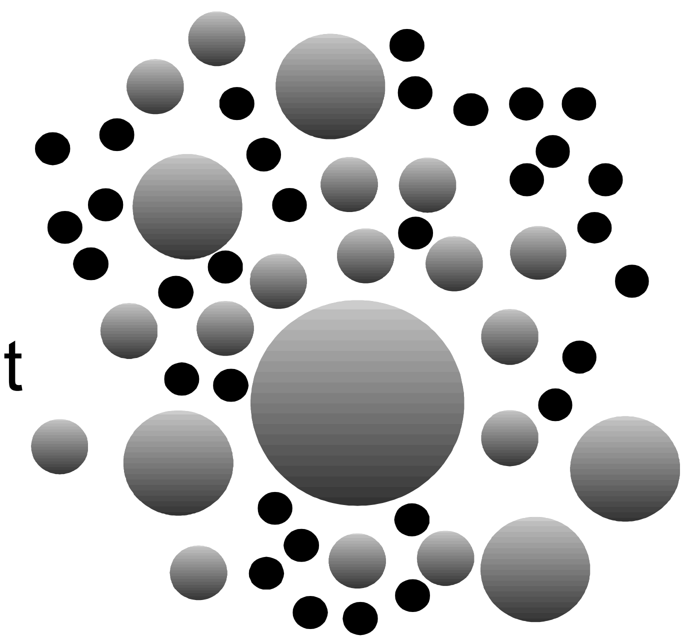
Non - Fractal

1 characteristic scale



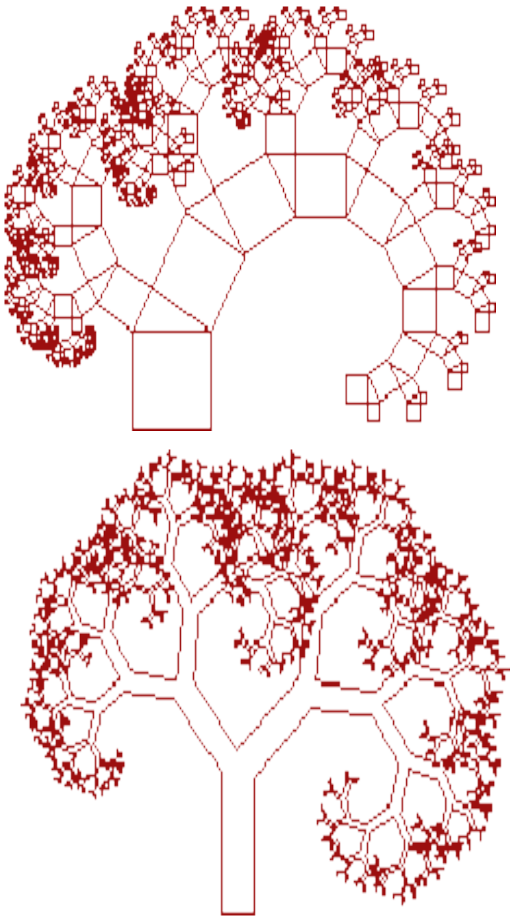
Fractal

many different scales



Larry S. Liebovitch

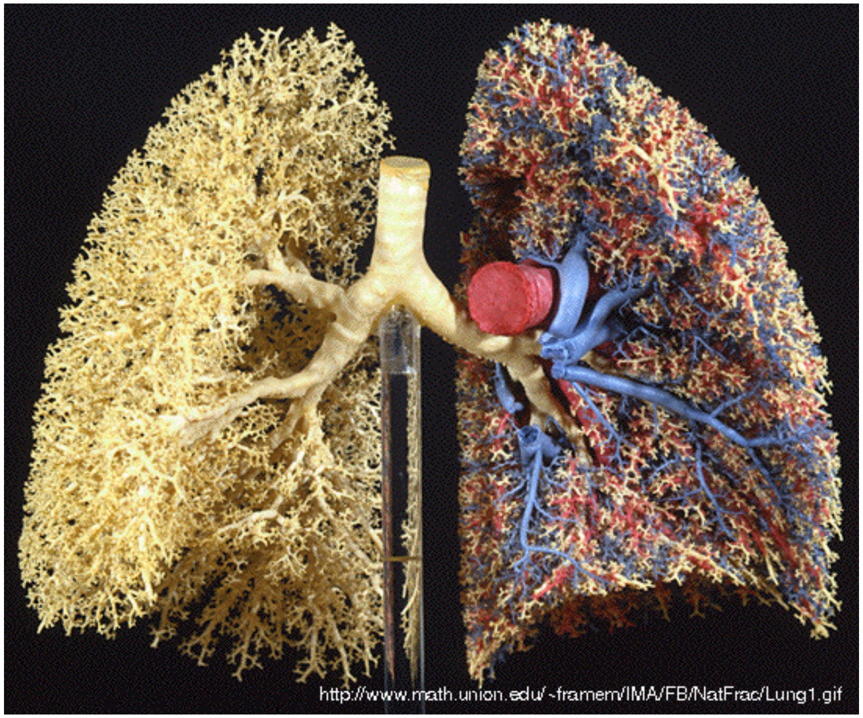
Deterministic nature of fractals



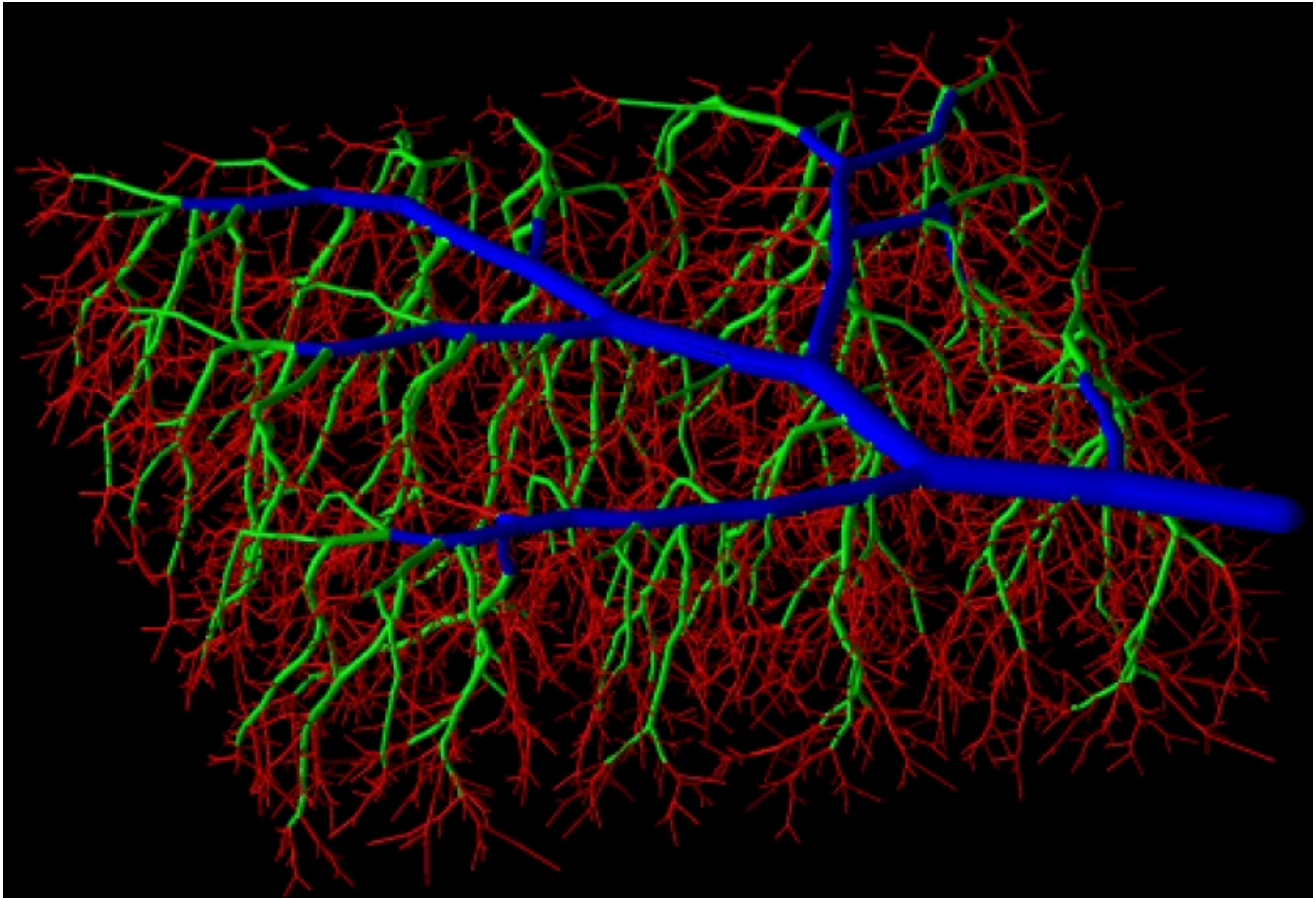
Pythagor tree

- Fractals are based on recursive mathematical formula: it can be predicted.
- Fractals are made to predict a complex and chaotic system deterministically.

Cardiovasular system



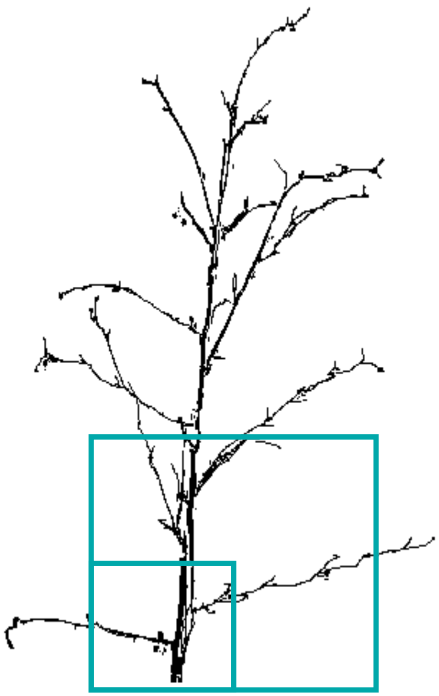
Similar to Pythagor tree



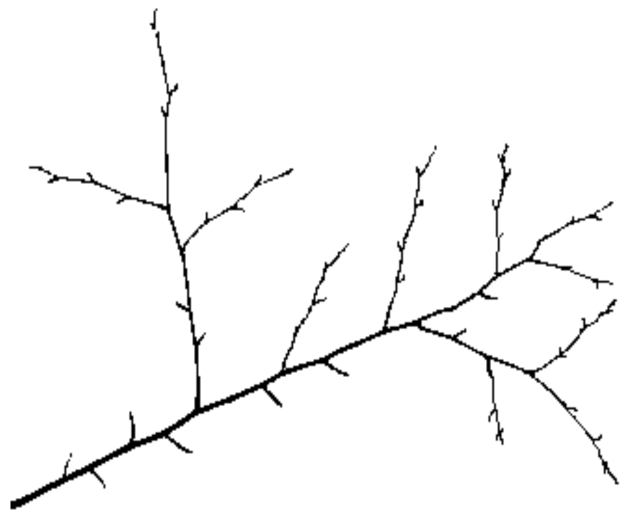
Arterial tree *Jun Zhang*

FD and evolution

Tree branching is related to evolution. There is a direct relationship between branching patterns and common ancestry (Bickel, 2000)



Domestic Birch



Wild Birch

Maple

Connection to tree graphs and graphs in general

FD Analysis Results

Type	Domestic	Wild
Birch	1.262	1.113
Cherry	1.647	1.459
Oak	1.648	1.279
Maple	1.378	1.358
Dogwood	1.656	1.456
Poplar	1.424	1.456

Euclidean Dimension (ED)

number of independent parameters that describes an object

- Dimension of an object measures the complexity of the object. 3D object is more complex than 1D object in general.
- Euclidean dimension: nonnegative integers 0,1,2,3,…
- Fractal dimension: 1.56, 2.49, 3.45,…

Fractal Dimension (FD)

- amount of variations in the object

Small FD = less jagged/complex

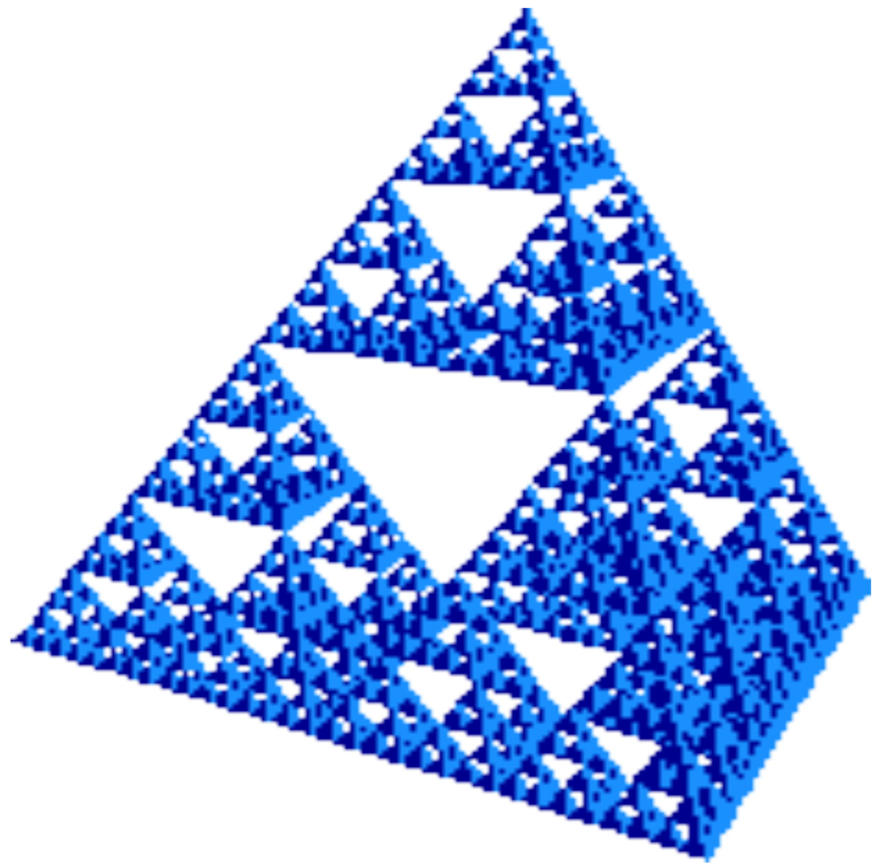
Large FD = more jagged/complex

Constructing Fractals

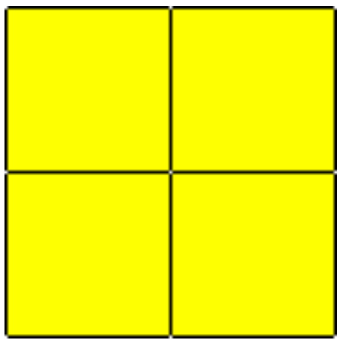
- Deterministic fractals are constructed by an iterative process with the initial configuration

Sierpinkskij lattice is constructed from a large triangle by recursively cutting smaller lattice.

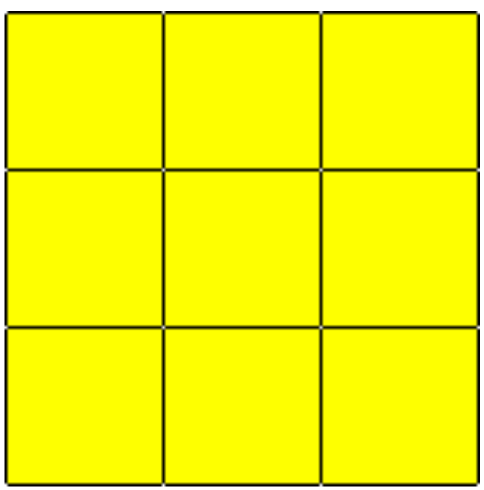
FD=ln3/ln2=1.58496…



Example of self-similarity



Dilate by a factor of k=2
N=4 copies of the self
similar original square.



Dilate by a factor of k=3
N=9 copies of the self
similar original square.

K=scale factor
N=number of copies.

FD=ln N/ln K

Measure of self-similarity

- For the square, we have

$$k^2 = N$$

- Alternately,

$$\log_k N = 2$$

Dimension of object

1D line embedded in 2D

Line segment

Original

Dilated

k = scale factor = 2

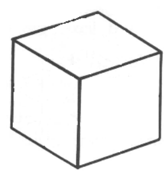
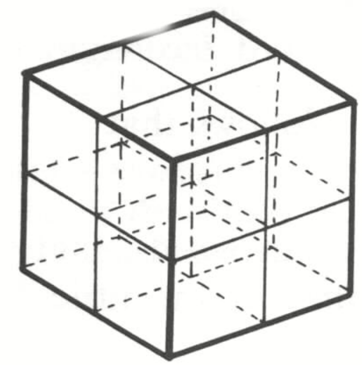
N = number of copies = 2

$$\log_k N = \frac{\ln N}{\ln k} = 1$$

Dimension of object

3D example

Cube

Original  Dilated 

k = scale factor = 2

N = number of copies of original = 8

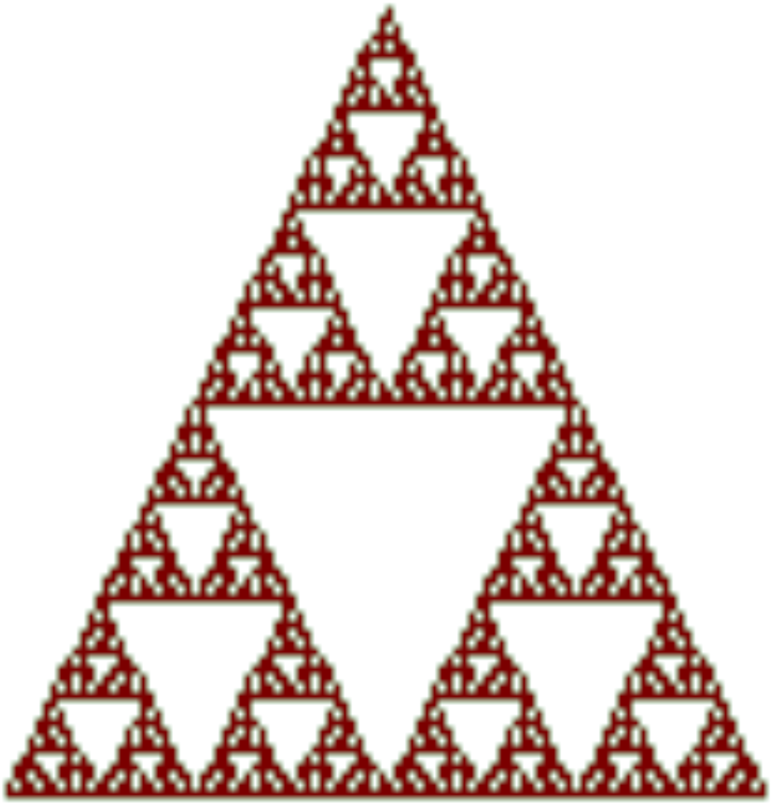
$$\log_k N = 3$$

Dimension of object

Fractal Dimension

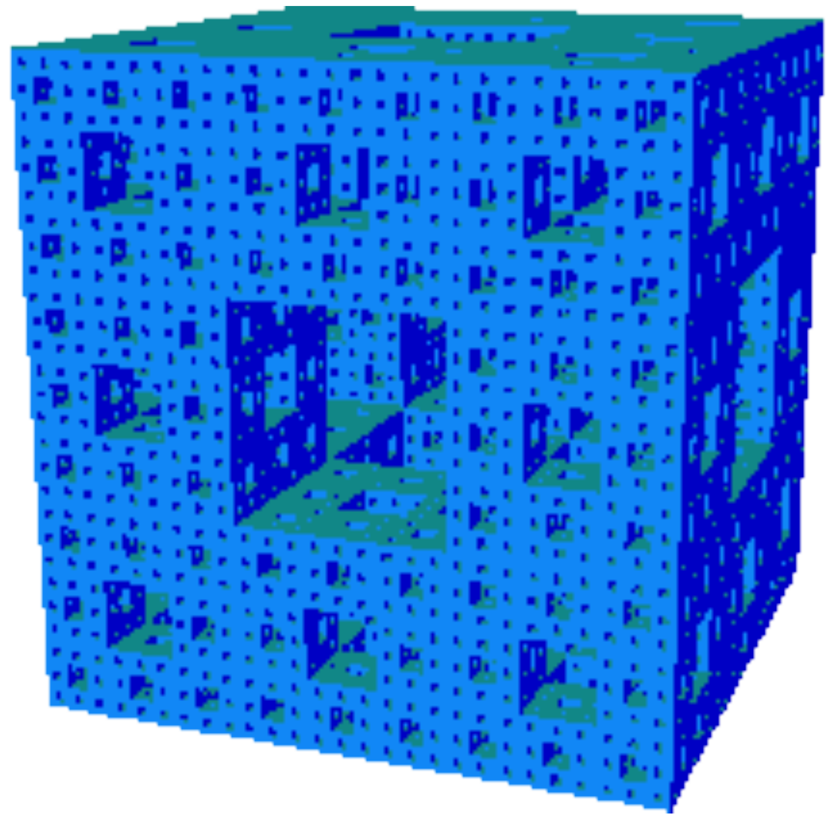
- $\log_k N$ measures the dimension of the object.
- This is the definition of the *dimension of a self-similar object.*
- We call this dimension as the **fractal dimension**.

Sierpinskij Triangle



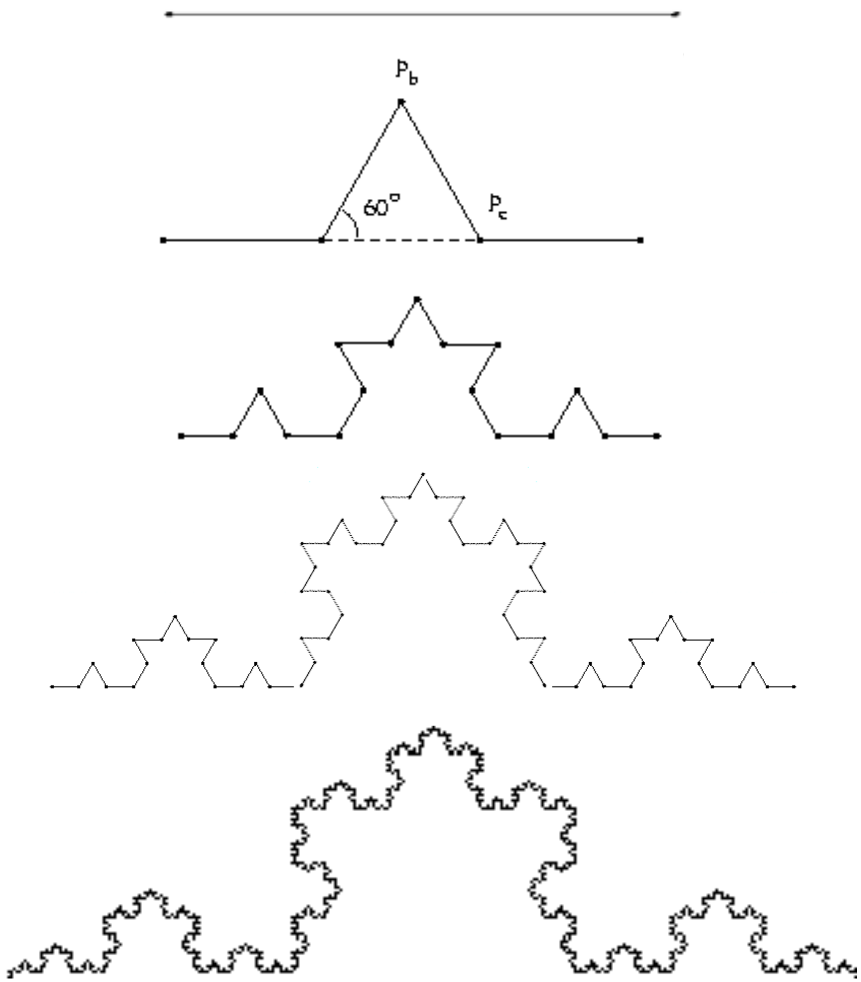
FD = ln 4/ln 2 = 2

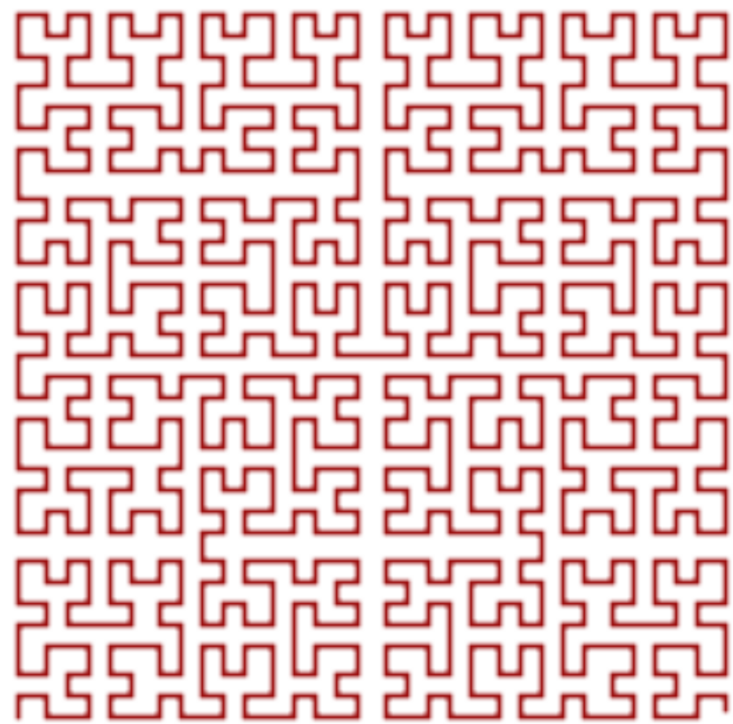
Sierpinskij Sponge



Koch curve

Hilbert curve

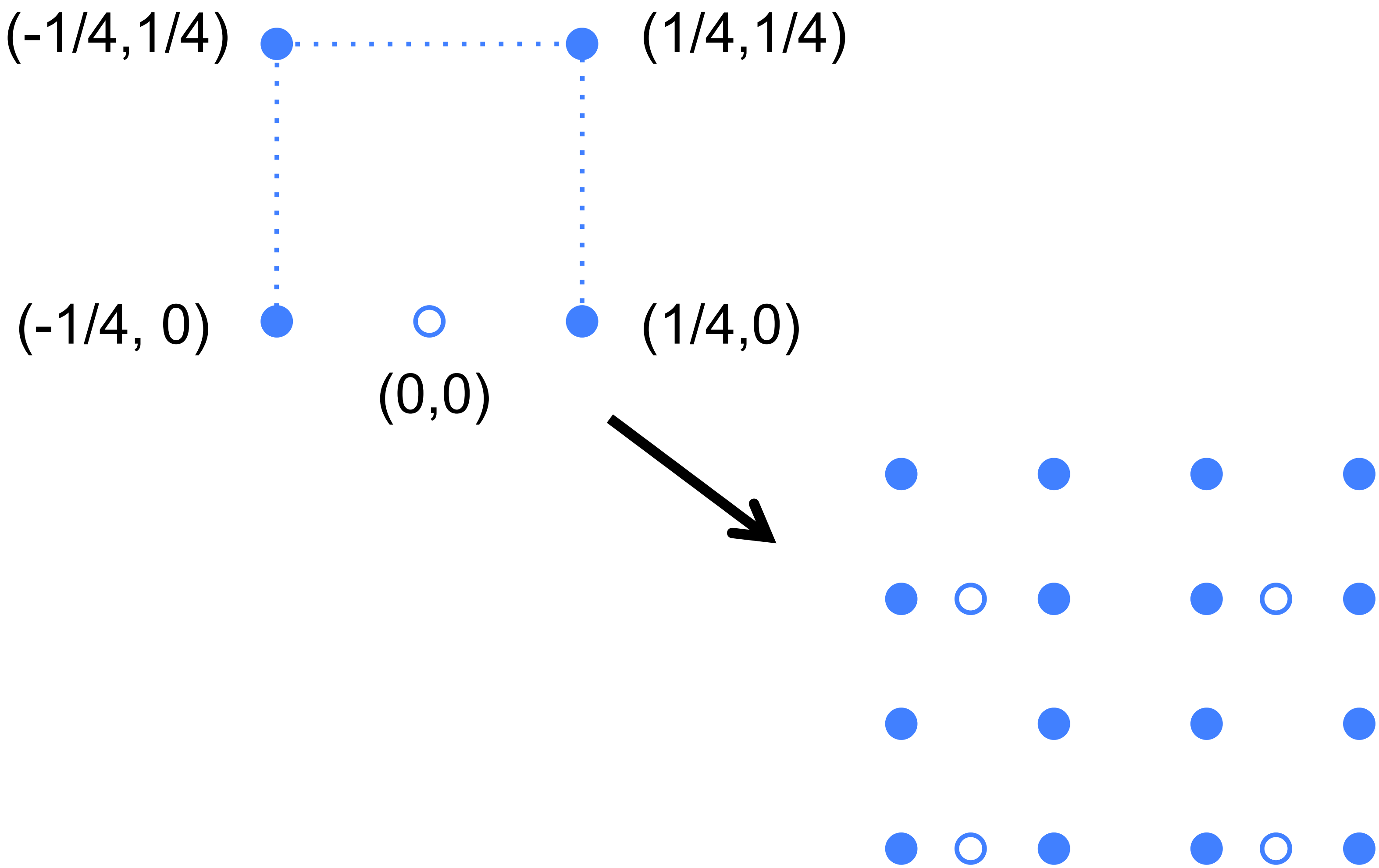




FD = ln 4/ln 3 = 1.261859….

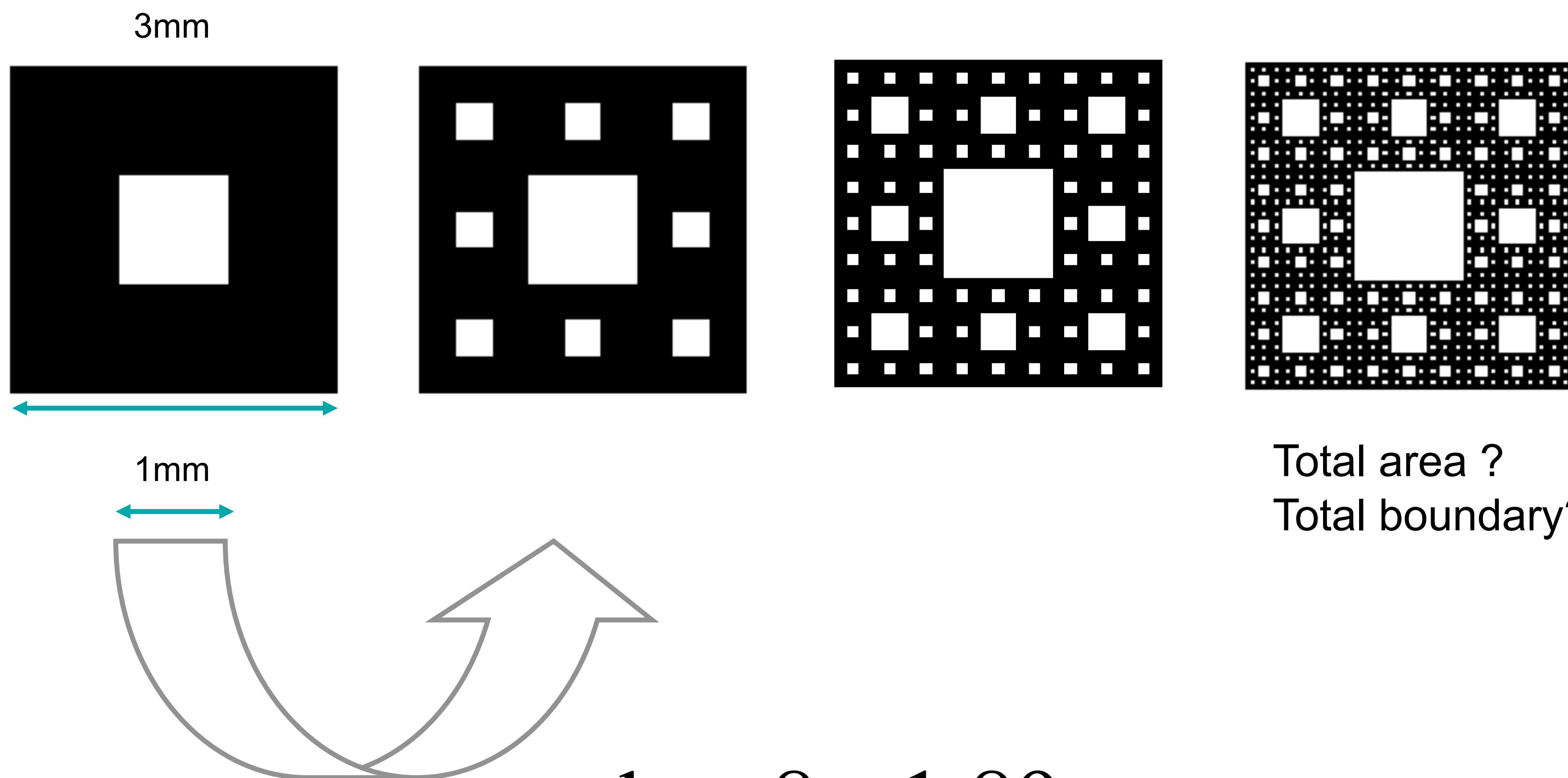
FD = ln 4/ln 2 = 2

Hilbert curve



Matlab demonstration

FD computation for Sierpinskij carpet



Total area ?
Total boundary?

$$\log_3 8 \approx 1.89$$

Estimating FD

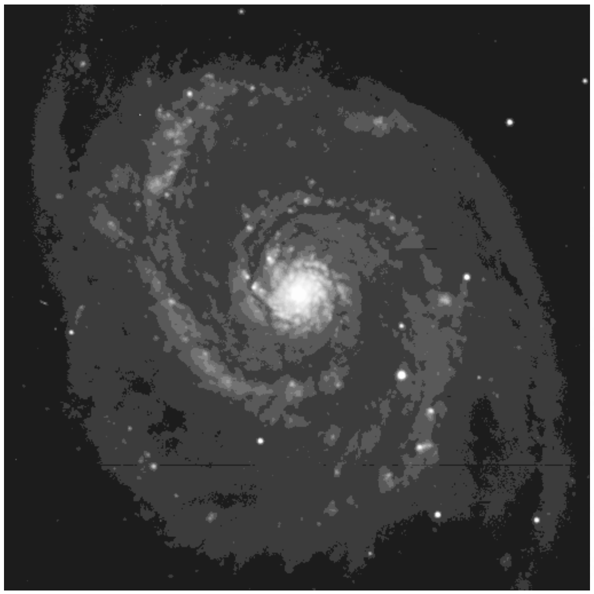
- FD can be explicitly computed on fractals that are mathematical given.

- In practice, the object of interest may not be a fractal but we can still estimate FD.

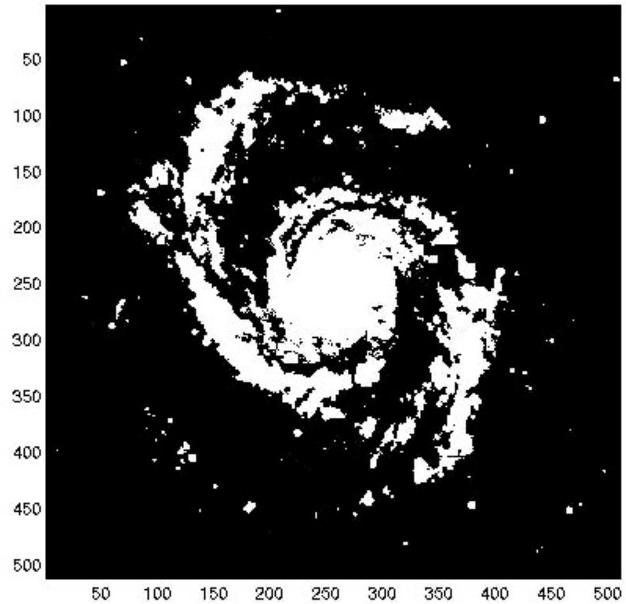
- Box-counting dimension (mandelbrot). The most widely used method

- Perimeter-area dimension (Hausdorff)

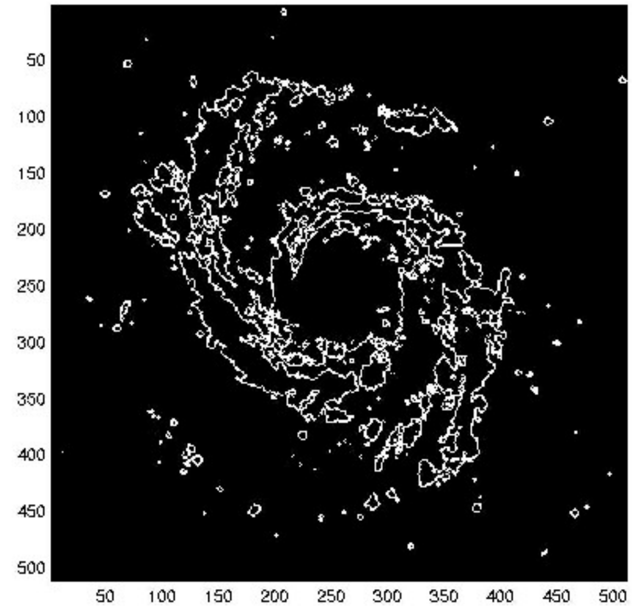
FD computation by box counting



Gray scale image

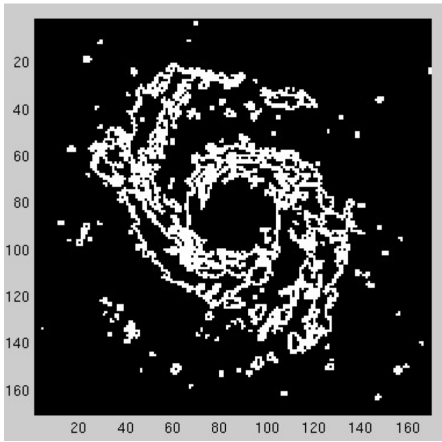


Binary image: pixels above a certain threshold are set to one

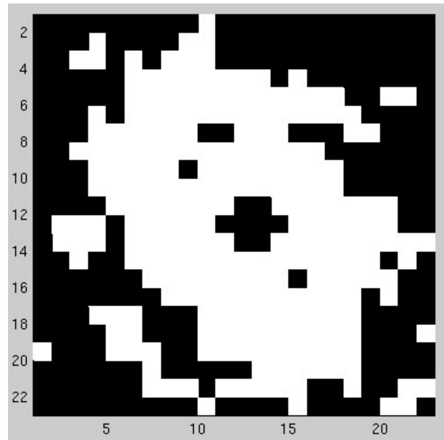


Boundary of the object is obtained

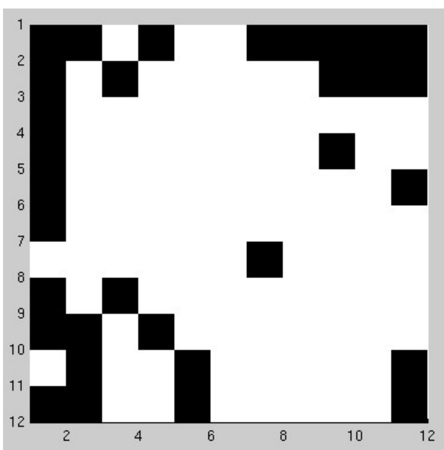
Box counting process



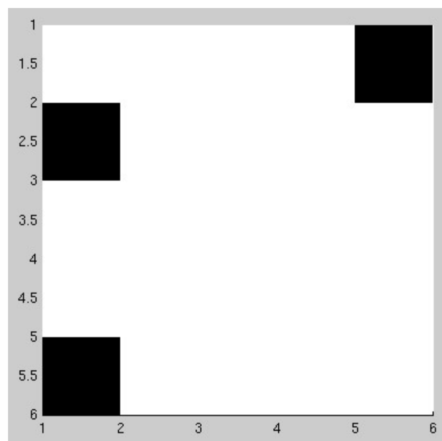
Box size 3 x 3



Box size 22 x 22



Box size 40 x 40



Box size 75 x 75

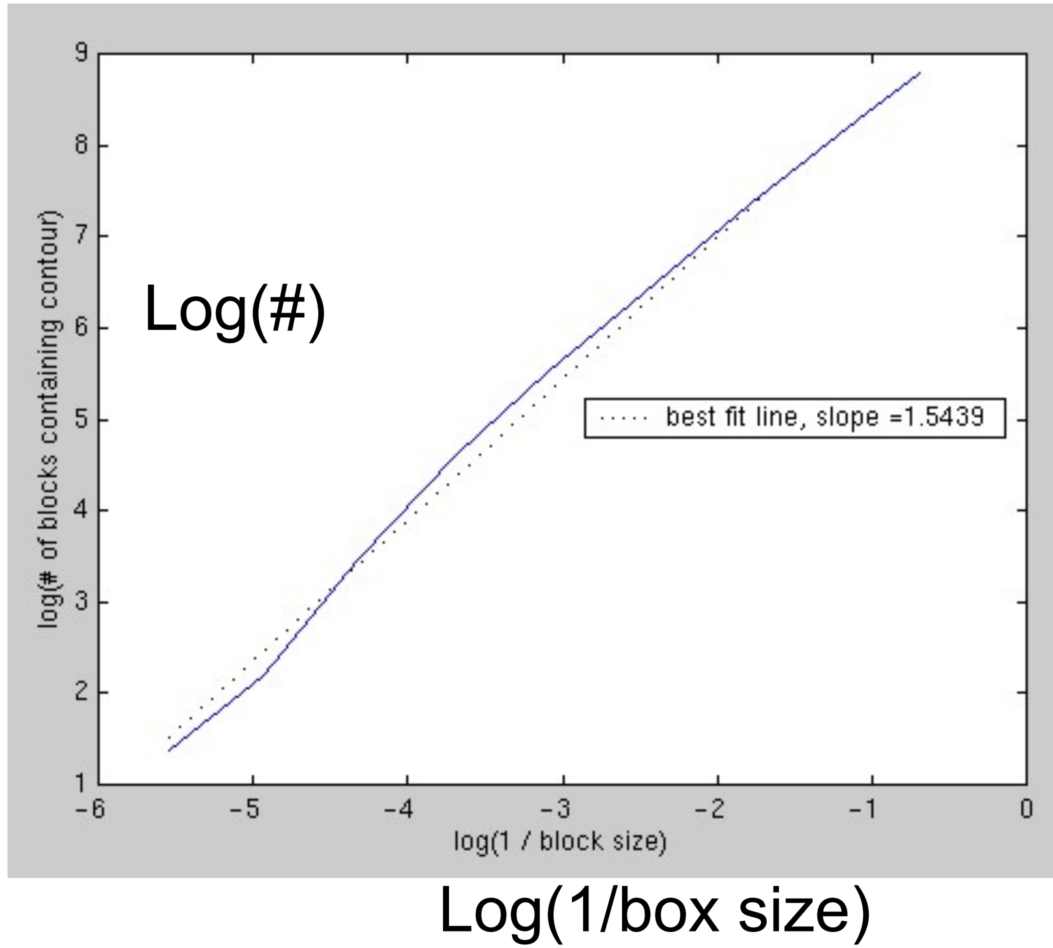
Break the image into boxes of a given size and count how many of those boxes contain the contour.

We perform this process for several different box sizes.

If a box contains the contour, it is colored white.

FD estimation

Box size (in pixels)	number of boxes containing the contour
2	6544
3	3897
6	1562
12	591
22	250
40	101
75	32
138	9
256	4



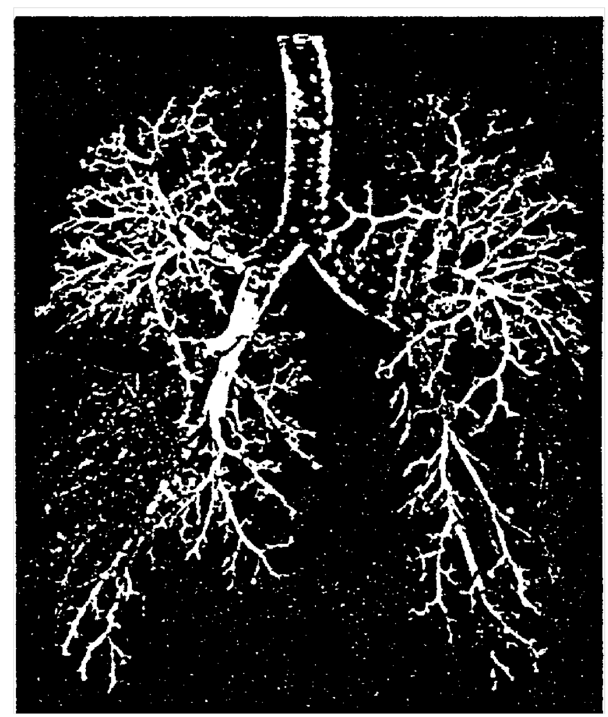
FD=slope =1.5439

blood vessels
in the retina

Family, Masters, and Platt 1989
Physica D38:98-103
Mainster 1990 Eye 4:235-241

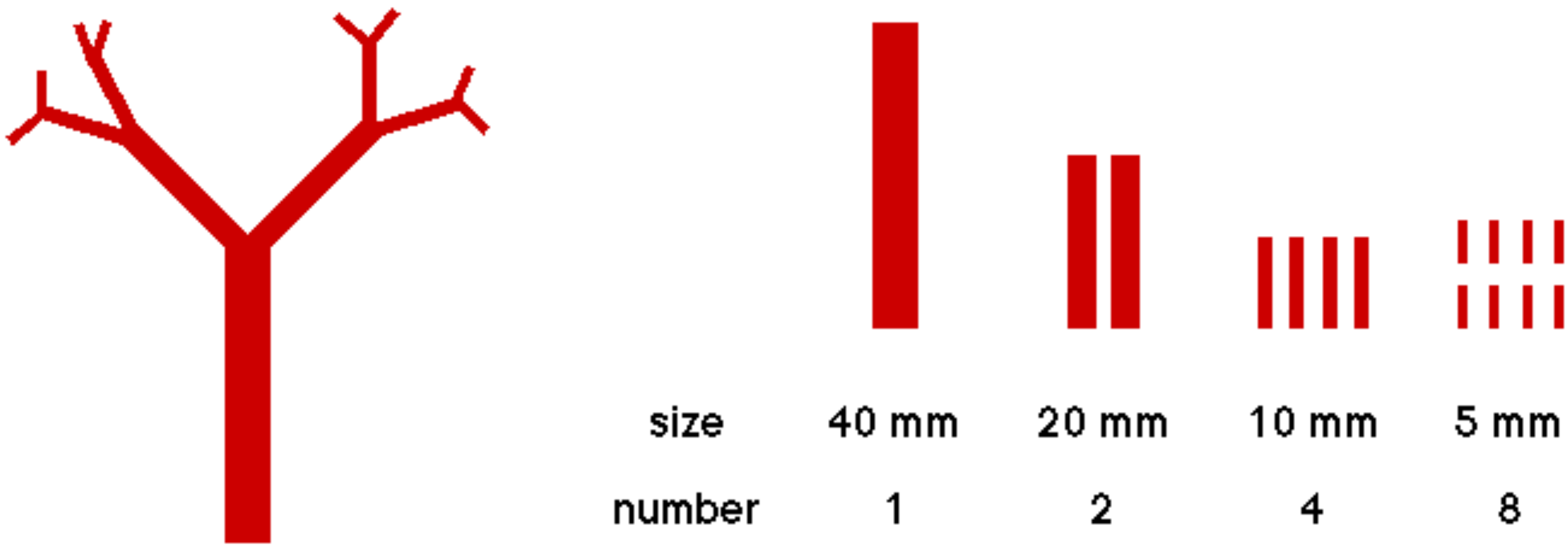
air ways
in the lungs

West and Goldberger 1987
Am. Sci. 75:354-365



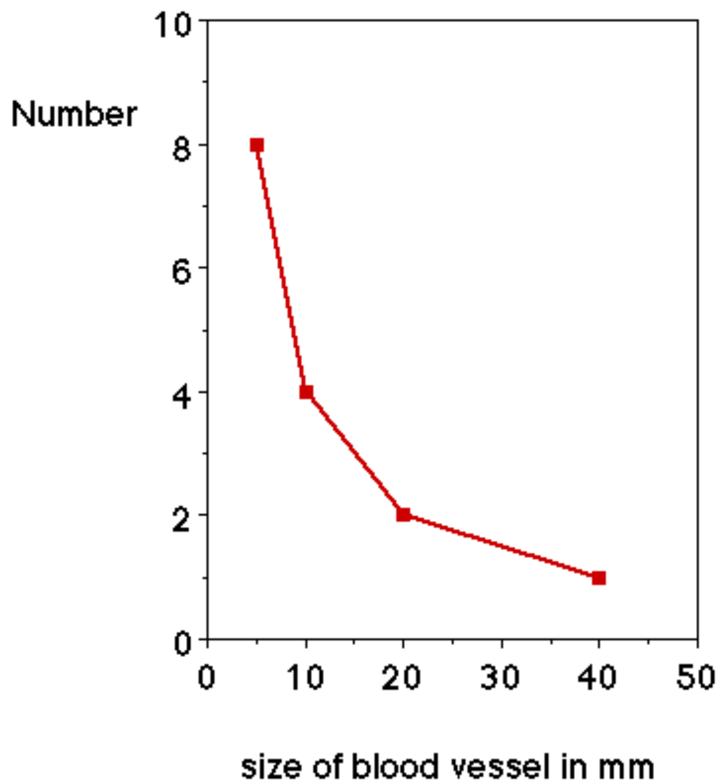
Probabilistic interpretation of FD

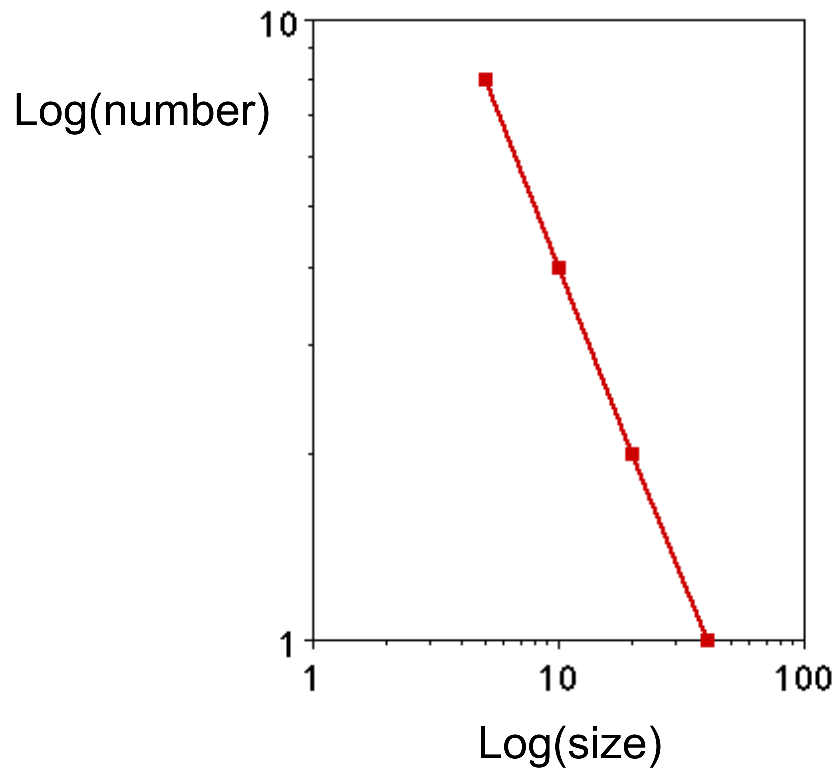
Blood Vessels in the Retina



PDF - Probability Density Function

HOW OFTEN there is THIS SIZE

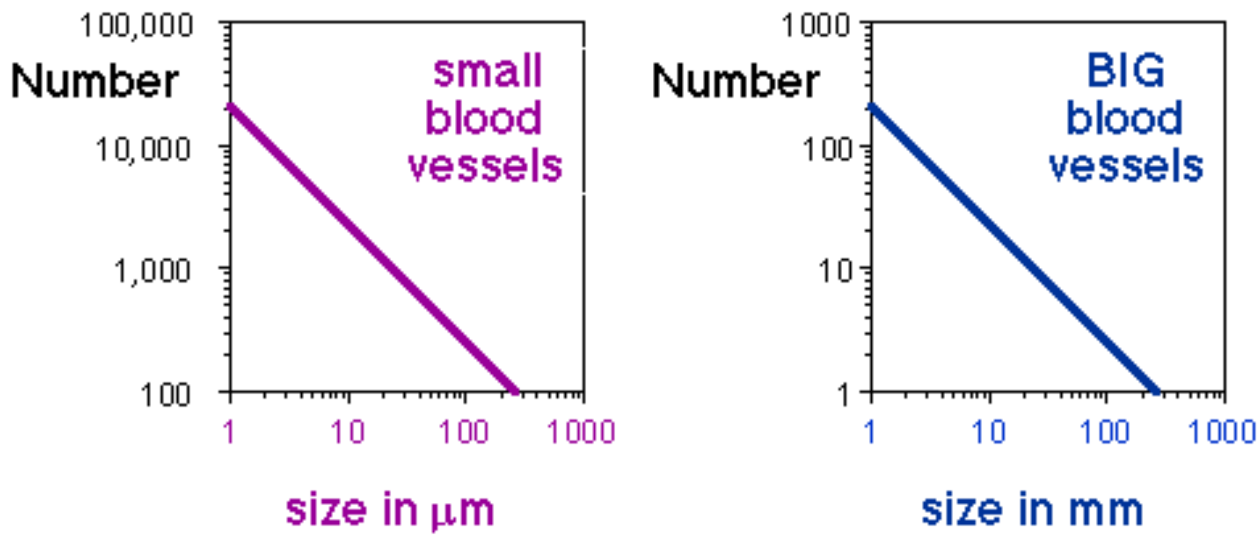




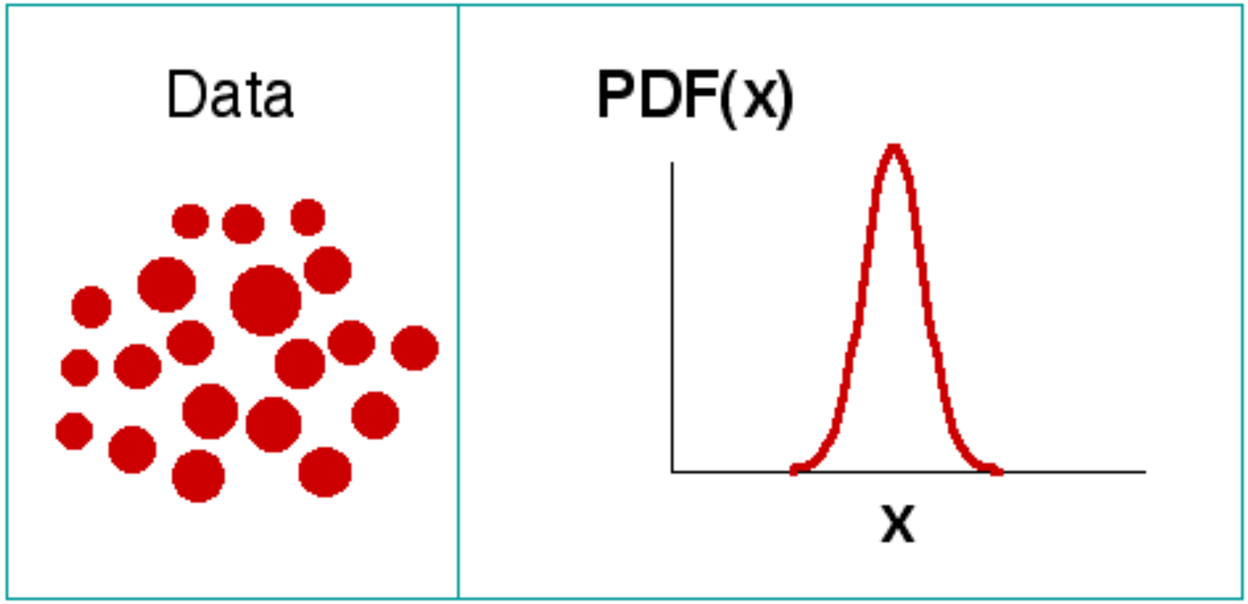
Straight line on log-log plot
= Power Law

Probabilistic self-similarity

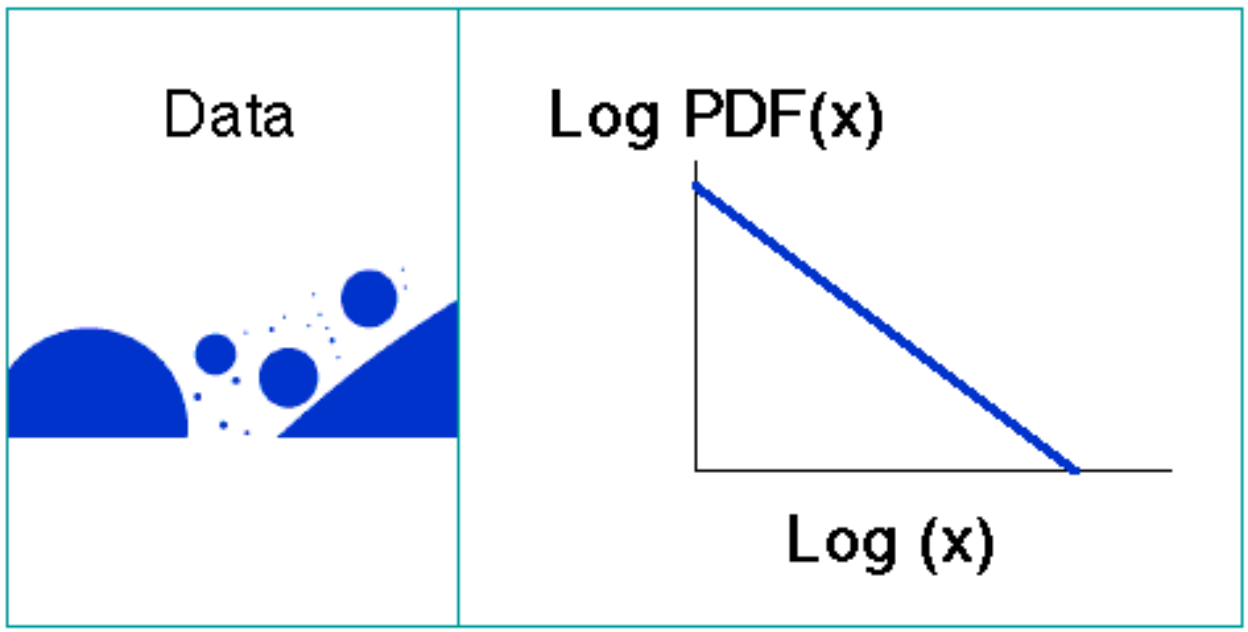
The statistics of the big pieces is the same as the statistics of the small pieces.



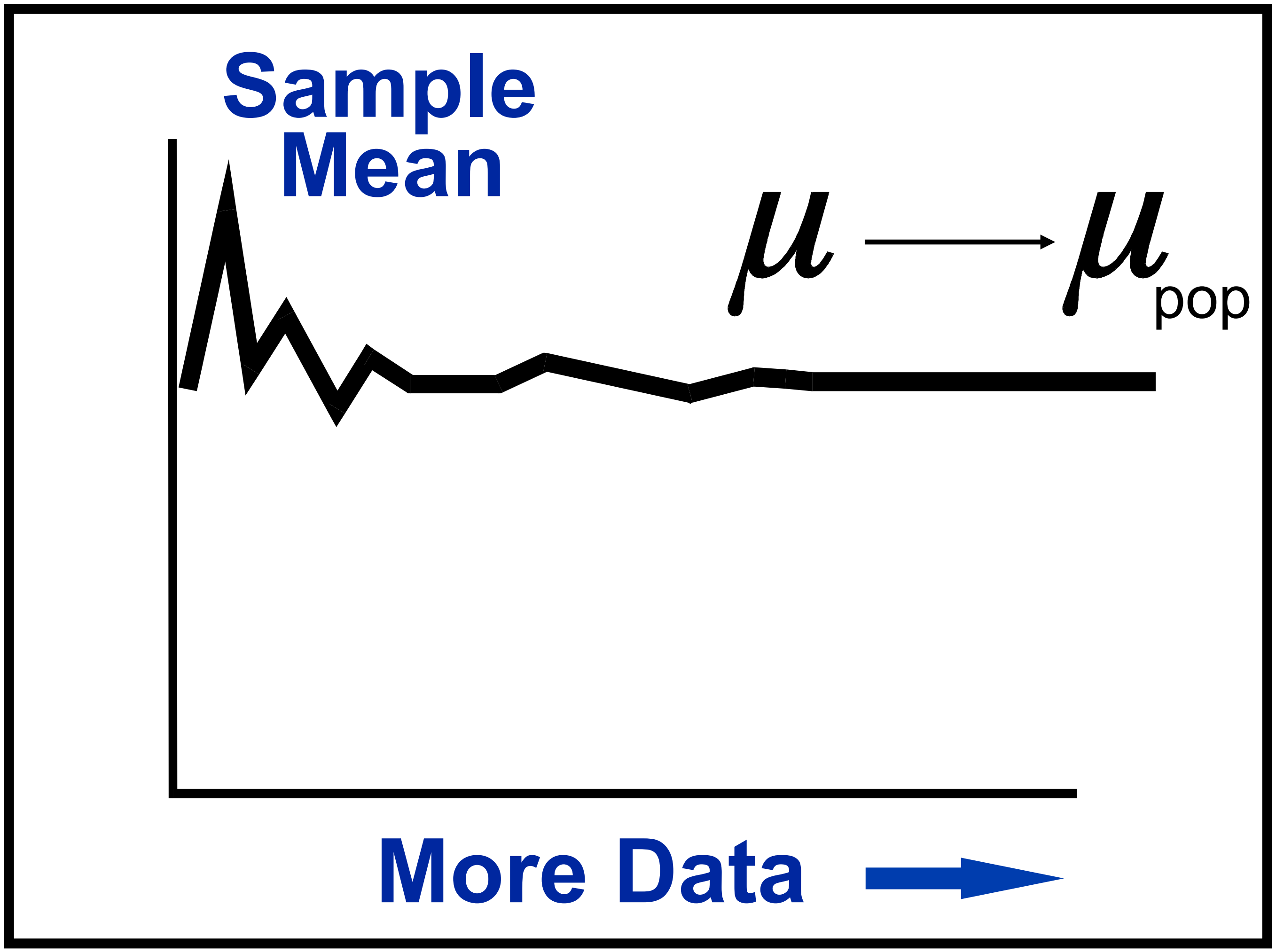
Not Fractal



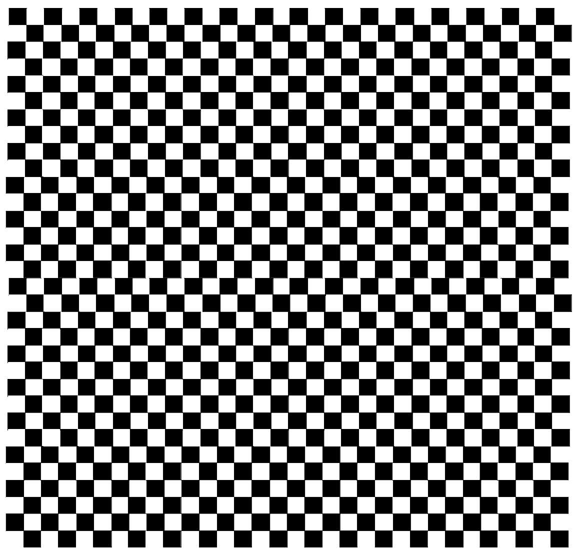
Fractal

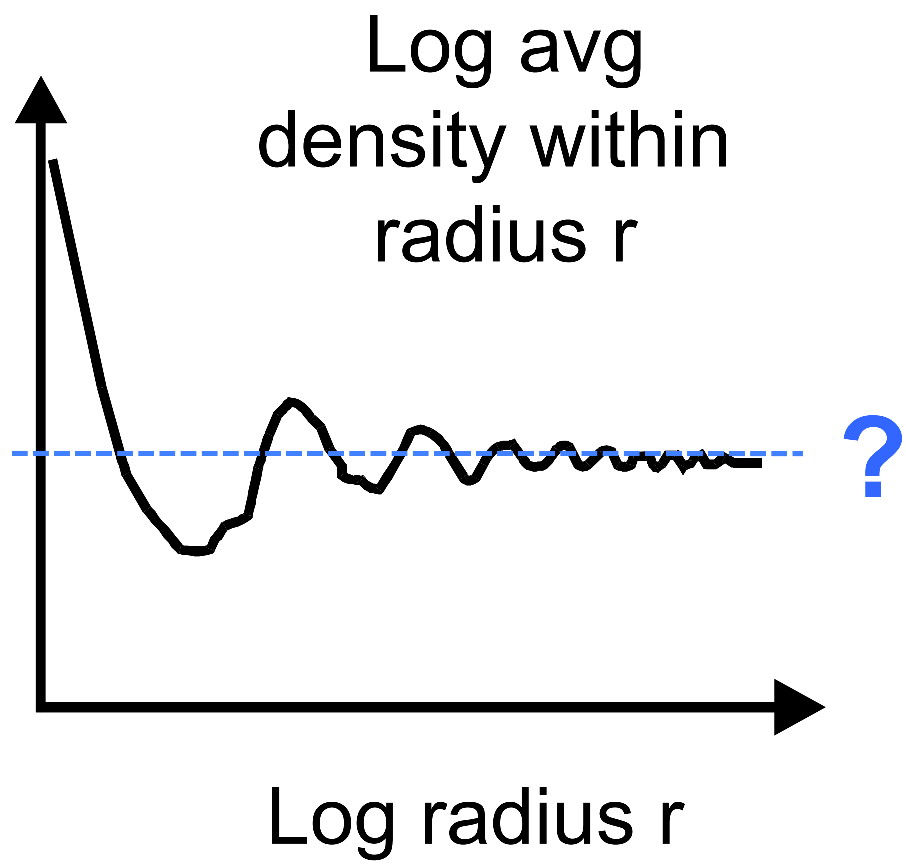


Non - Fractal

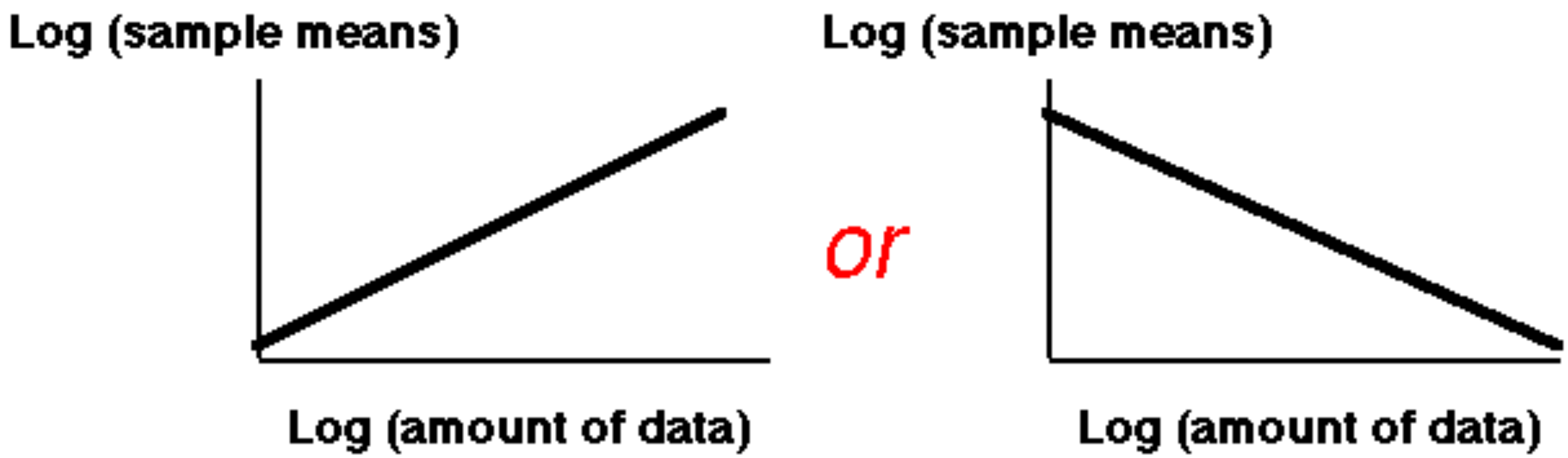


Non-Fractal





For fractals, the Average depends on the amount of data analyzed.



Ordinary Coin Toss

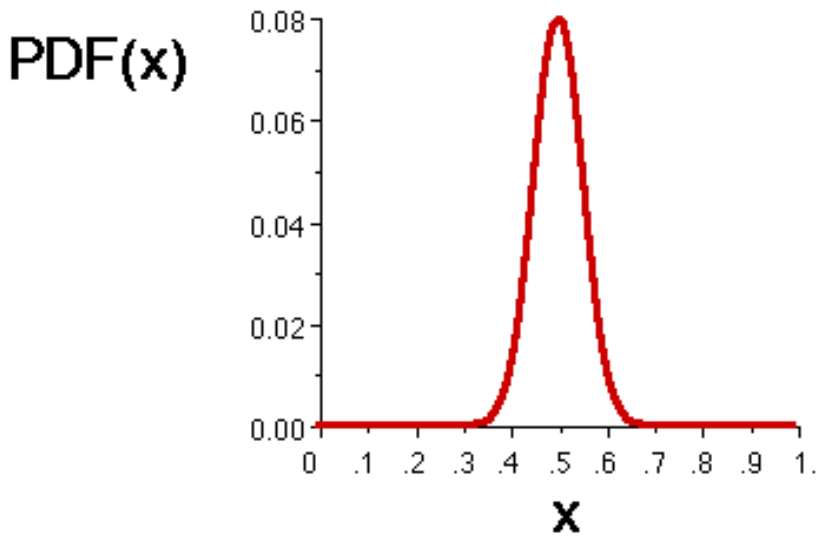
Toss a coin. If it is a tail win $0, If it is a head win $1.

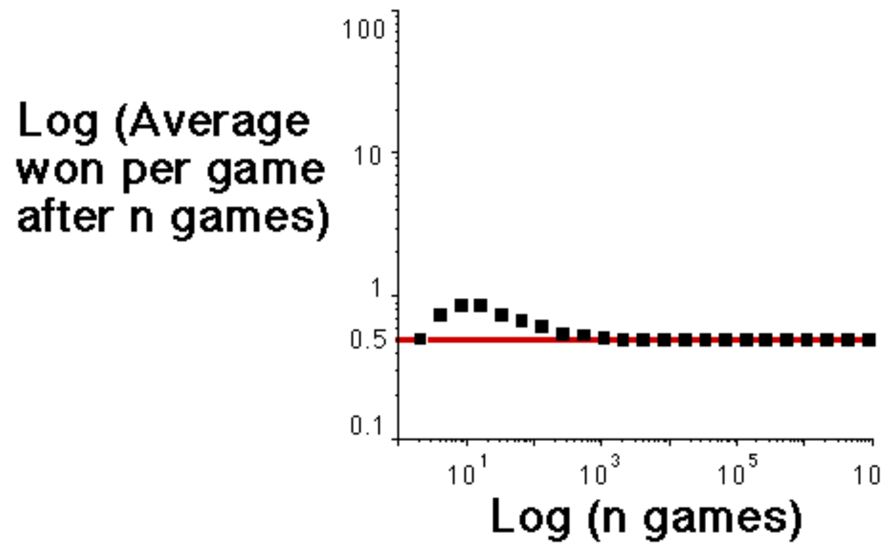
The average winning is $0.5

$$\mu \longrightarrow \mathbf{1/2}$$

Non-Fractal

Ordinary Coin Toss





St. Petersburg Game (Niklaus Bernoulli)

Toss a coin. If it is a head win \$2, if not, keep tossing it until we obtain a head.

If this occurs on the N-th toss we win $\$2^N$.

With probability 2^{-N} we win $\$2^N$.

H	\$2
TH	\$4
TTH	\$8
TTTH	\$16

The average winnings are:

$$2^{-1}2^{1} + 2^{-2}2^{2} + 2^{-3}2^{3} + \ldots =$$

$$1 + 1 + 1 + \ldots = \infty$$

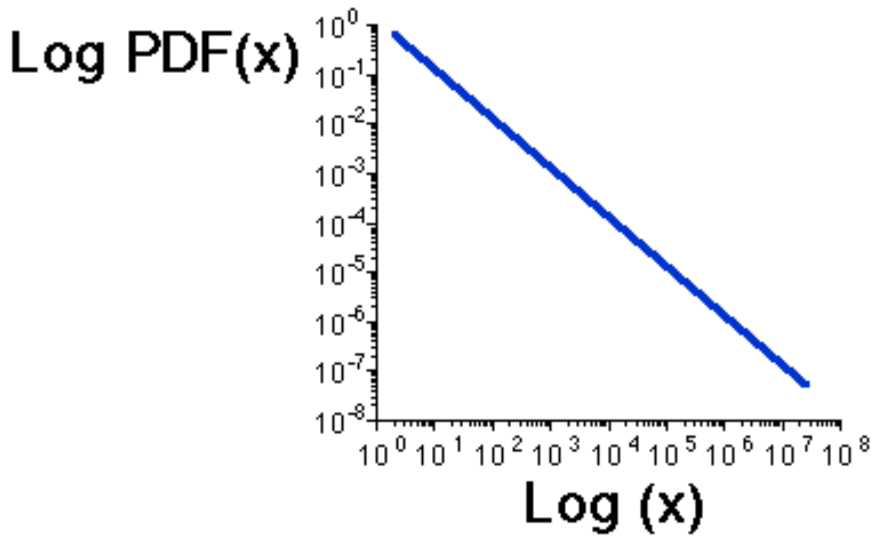
μ

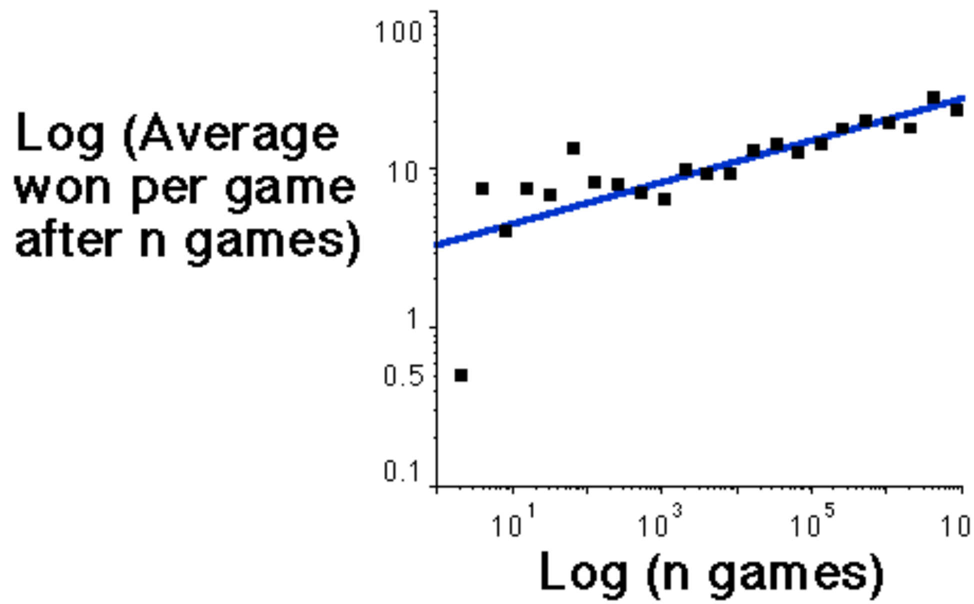
$\downarrow$

∞

Fractal

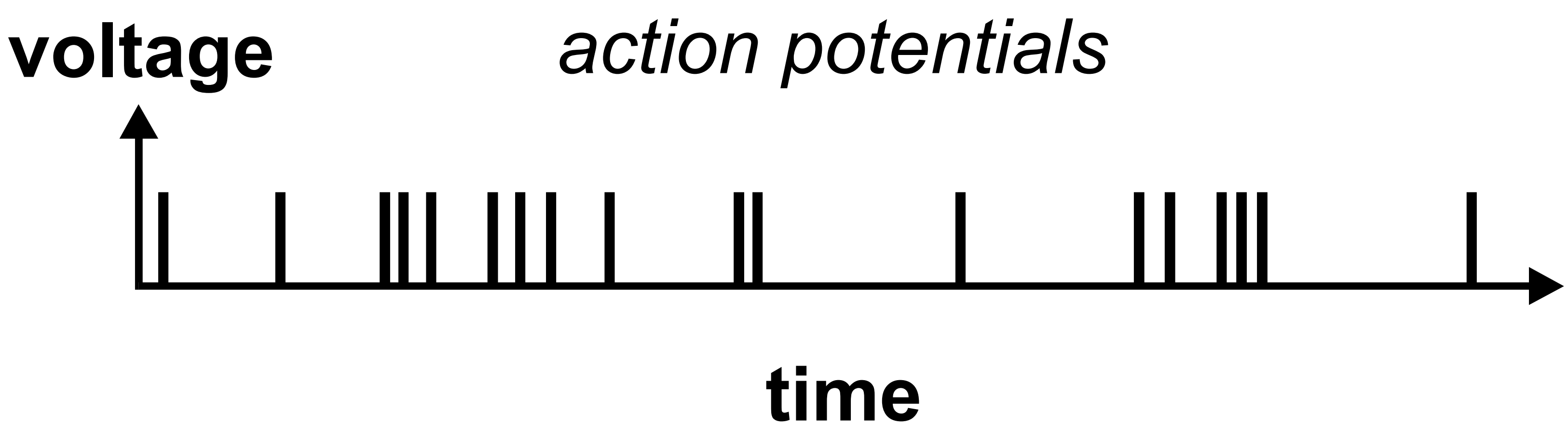
St. Petersburg Game (Niklaus Bernoulli)





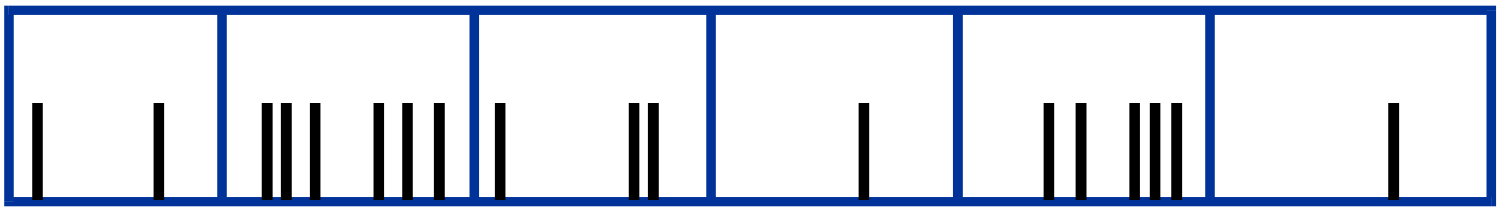
Electrical Activity of Auditory Nerve Cells

Teich, Jonson, Kumar, and Turcott 1990
Hearing Res. 46:41-52



Divide the record into time windows:

Count the number of action potentials in each window:



2 6 3 1 5 1

Firing Rate = 2, 6, 3, 1, 5,1

Repeat for different lengths of time windows:

8 4 6

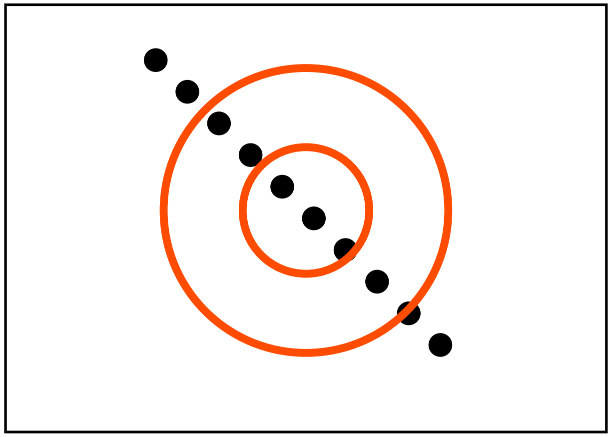
Firing Rate = 8, 4, 6

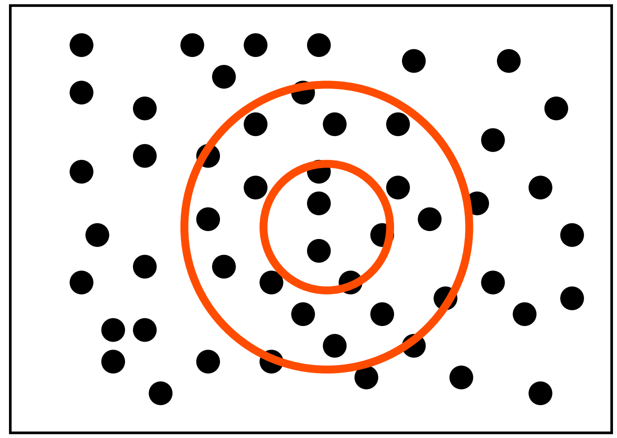
Dimension of point cloud data

•FD can be taken as the approximation to the intrinsic dimension (D_i) of data, which can be different from the embedding dimension (D_e).

$$count\left(x, x' : \|x - x'\| < k\right) = k^{D_i}$$

$$D_i = \frac{\ln count\left(x, x' : \|x - x'\| < k\right)}{\ln k}$$





- line in a plane:
  - $D_e=2$
  - $D_i=1$
  - FD $\approx$ 1

- uniform dist. in a plane:
  - $D_e=2$
  - $D_i=2$
  - FD $\approx$ 2

Read camastra.2002.PAMI….

Unfortunately, the box-counting dimension can be computed only for low-dimensional sets because the algorithmic complexity grows exponentially with the set dimension. Therefore, in our opinion, a good substitute for the box-counting dimension can be the *correlation dimension* [11]. Due to its computational simplicity, the correlation dimension is successfully used to estimate the dimension of attractors of dynamical systems. The correlation dimension is defined as follows: let $\Omega = \mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_N}$ be a set of points in $\mathbb{R}^n$ of cardinality N. If the *correlation integral* $C_m(r)$ is defined as:

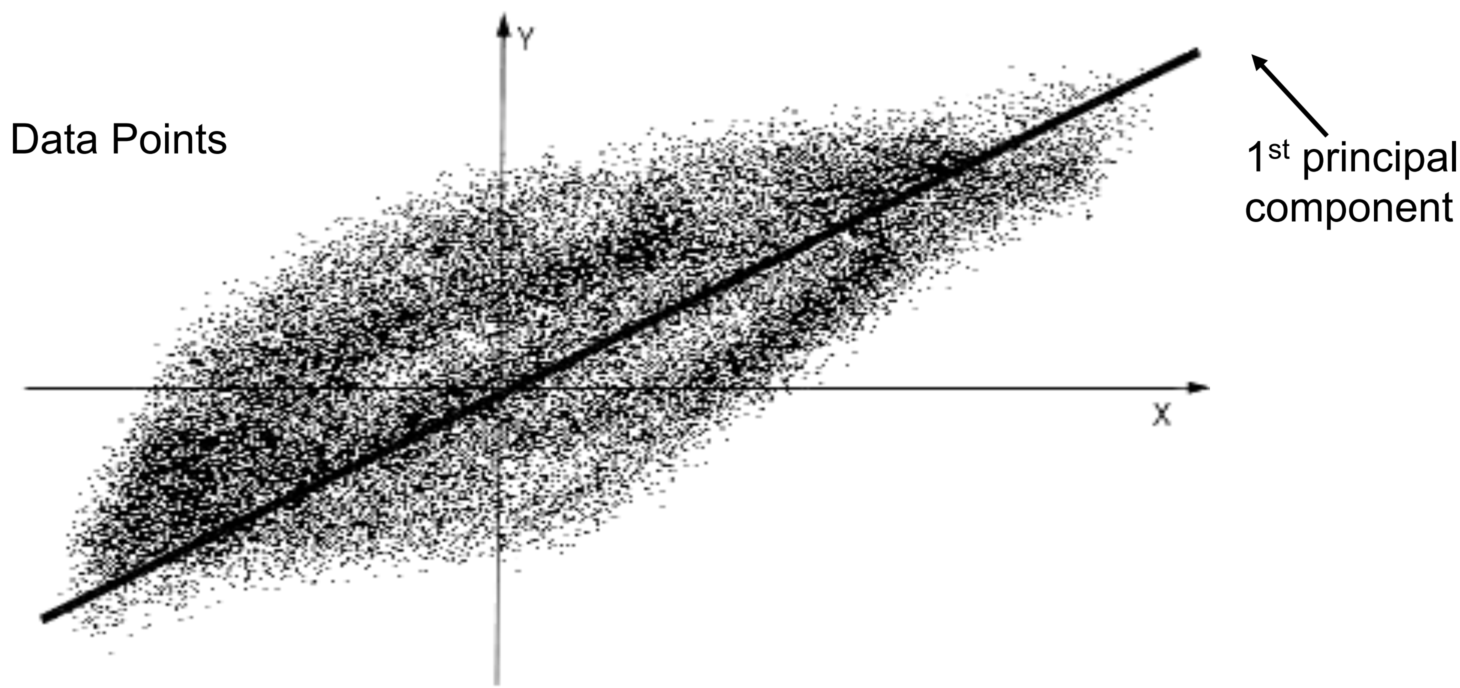
$$C_m(r) = \lim_{N \to \infty} \frac{2}{N(N-1)} \sum_{i=1}^{N} \sum_{j=i+1}^{N} I(\|\mathbf{x_j} - \mathbf{x_i}\| \leq r), \qquad (2)$$

where I is an *indicator function,*[1] then the *correlation dimension* D of Ω is:

$$D = \lim_{r \to 0} \frac{\ln(C_m(r))}{\ln(r)}. \qquad (3)$$

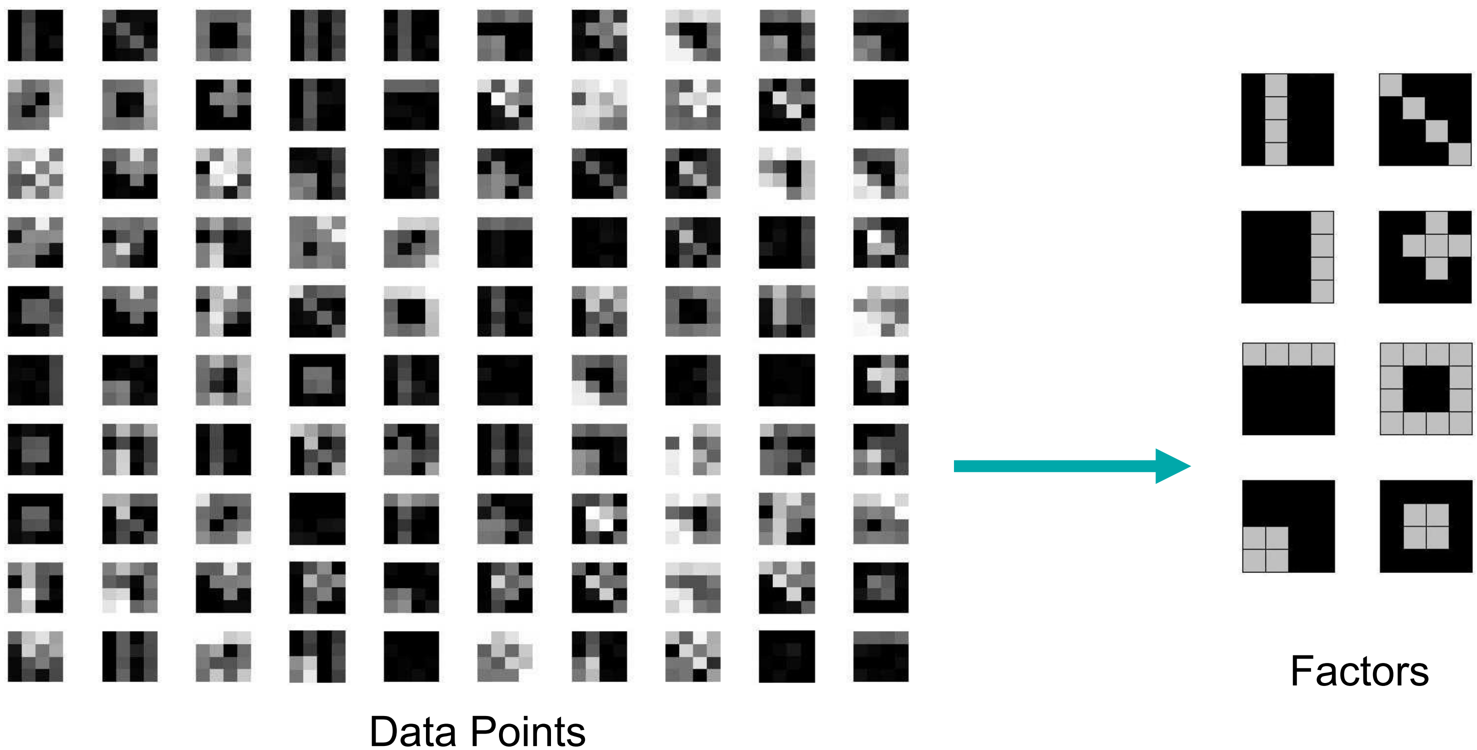
Read camastra.2002.PAMI….

FD and PCA



FD and the number of significant principal component ?

FD and Factor Analysis



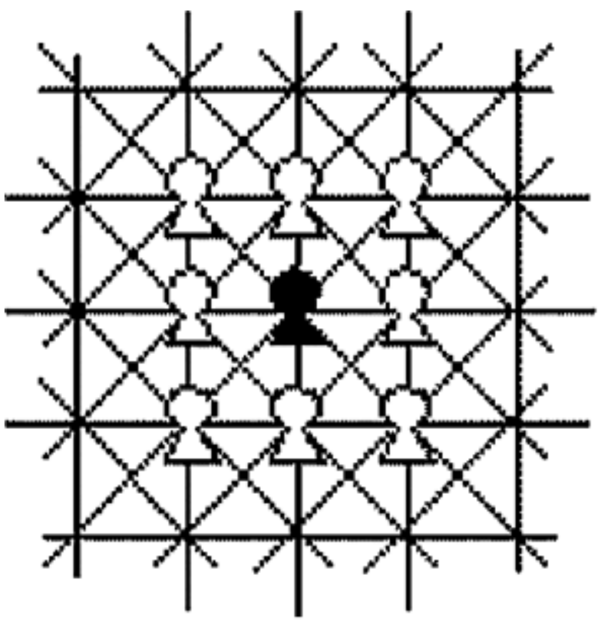
FD and the number of significant factors ?

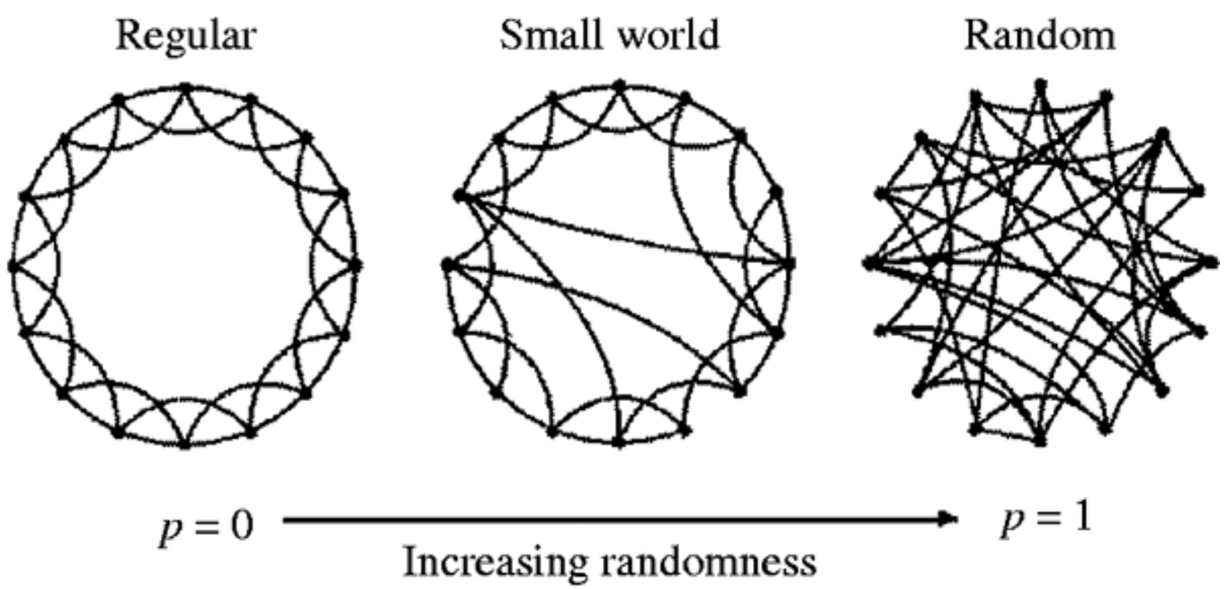
Are brain networks self-similar?

Small-world network

•Six degrees of separation in social networks.

•One can reach a given node from another one with a small number of steps.





Source: Watts (1999)

Small-world. The *distance* between two nodes in a network is the number of edges in a shortest path connecting them. If most nodes can be connected in a very small number of steps, the network is said to be *small-world.* Let $\mathfrak{l}$ be the shortest distance between two nodes and $\mathfrak{n}$ is the number of nodes in a graph. Then small-worldness is mathematically expressed as (Song *et al.*, 2005):

$$\mathbb{E}\mathfrak{l} \sim \ln \mathfrak{n}. \tag{9.32}$$

$\mathbb{E}\mathfrak{l}$ is sometime called the diameter of the network. However, in general, the *diameter* of a network usually means the maximum $\mathfrak{l}$ (longest geodesic path) (Newman, 2003). The relation (9.32) links the over all size of the graph to the number of nodes. (9.32) implies that the small-world networks are not self-similar, since self-similarity requires a power-law relation between $\mathfrak{l}$ and $\mathfrak{n}$. However, via a scale-invariant renormalization procedure, one can show diverse complex networks are in fact self-similar (Song *et al.*, 2005).

Small-worldness

$$E(l) \propto \ln n$$

$$n \propto e^{E(l)}$$

Contradiction

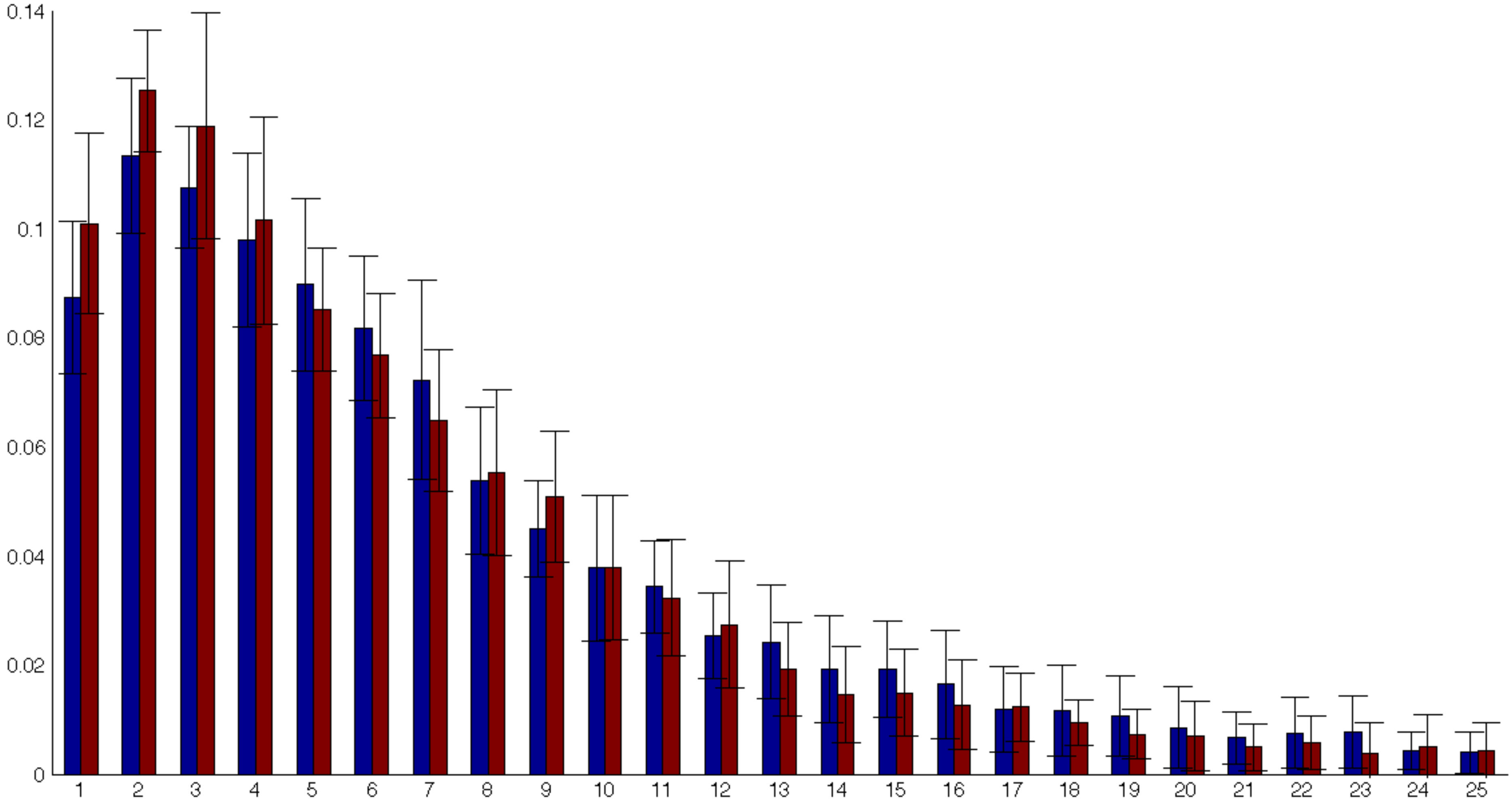
Self similarity

$$n = E(l)^{FD}$$

$$\frac{\ln n}{\ln E(l)} = FD$$

Degree distributions

red: autism
blue: control



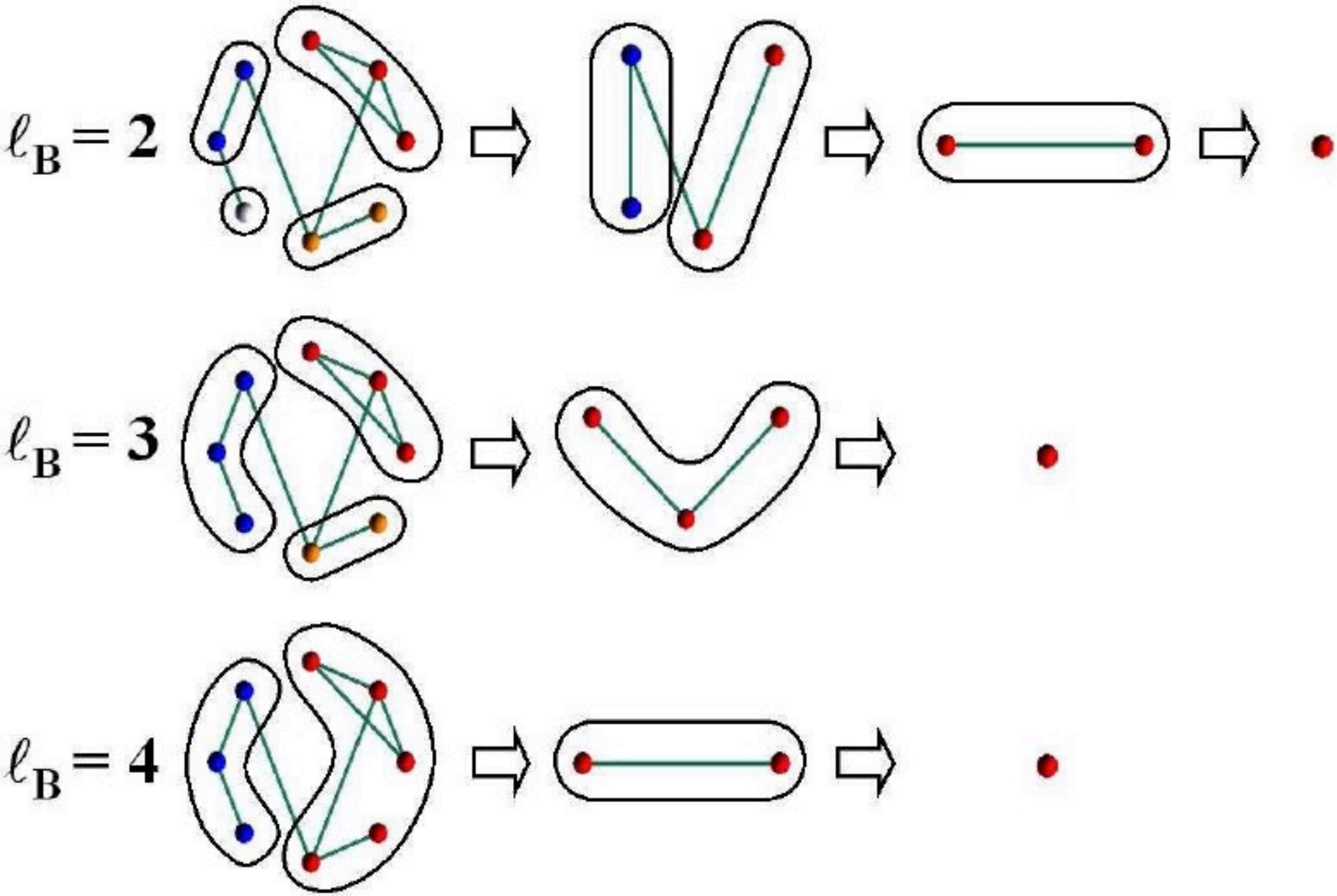
Scale-free network

The *degree distribution* P(k), probability distribution of the number of connecting edges in each node, can be represented by a power-law with a degree exponent γ usually in the range $2 < \gamma < 3$ for diverse networks (Bullmore and Sporns, 2009; Song *et al.*, 2005):

$$P(k) \sim k^{-\gamma}.$$

Such networks exhibits gradual decay of tail regions (heavy tail) and are said to be *scale-free*. In a scale-free network, a few hub nodes hold together

Scale invariant renormalization process: *song.2005.pdf*



14 control subjects   14 random graphs