Computational Methods in NeuroImage Analysis

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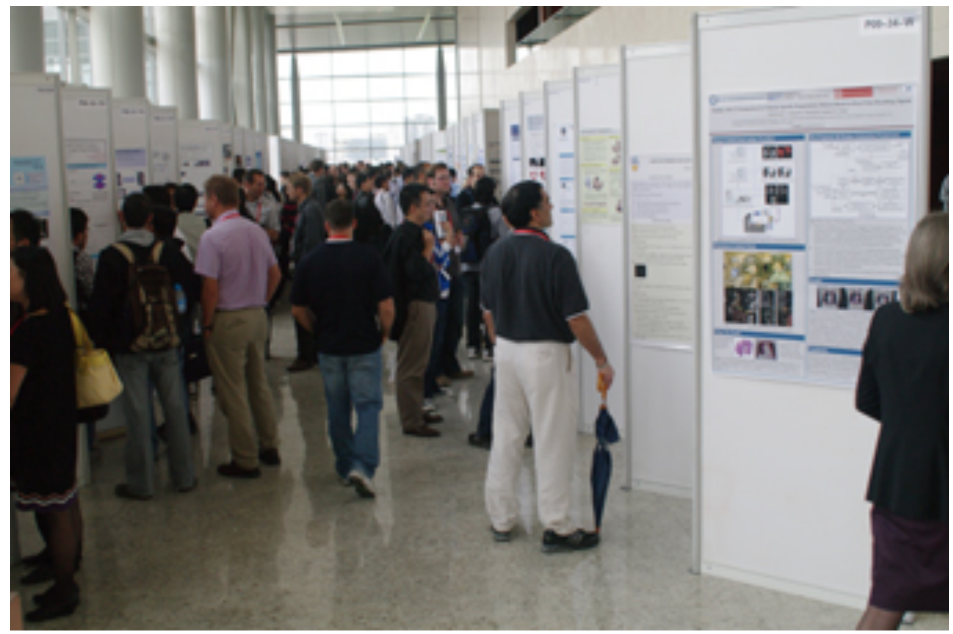
Lecture 4

Iterative Residual Fitting (IRF) Algorithm

September 24, 2010

MICCAI 2010

13th International Conference on Medical Image Computing and Computer Assisted Intervention



Important medical imaging conferences

MICCAI 2011
Venue: Toronto Canada
Conference Dates: Sept 18-22, 2011
Submission deadline: March

MICCAI 2012
Venue: Nice Acropolis, Nice, Côte d'Azur, France
Conference Dates: Oct 1-5, 2012
Submission deadline: March

Deadline for submission of 4-page paper: October 28, 2010

9th IEEE International Symposium on
Biomedical Imaging (ISBI 2012)
5/2/2012 to 5/5/2012
Barcelona , Spain

Information Processing in Medical Imaging (IPMI) 2011

July 3-8, 2011
Monastery Irsee
Germany (Bavaria)

Submission deadline
December 13, 2010

17th Human Brain Mapping Meeting

OHBM 2011

June 26-30, 2011
Centre des Congres de Quebec
Quebec City, Canada

Submission deadline
January

Weighted Fourier Analysis

Read Chapter 7

Related works in neuroanatomy

Surface model, parameterization → Surface registration → Surface data smoothing → Multiple comparison correction

Spherical harmonic descriptors
Guido Gerig
Martin Styner
Li Shen

PDE
Paul Thompson
Michael Miller

diffusion smoothing (NeuroImage 2003 CVPR, 2003)
heat kernel smoothing (NeuroImage, 2005)

Random field theory
Keith Worsley
Jonathan Taylor

Unified framework: Weighted Fourier Analysis (IEEE -TMI, 2007)

SPHARM representation

Framework for the Statistical Shape Analysis of Brain Structures using SPHARM-PDM

Release 1.00

Martin Styner[1,2], Ipek Oguz[1], Shun Xu[1], Christian Brechbühler, Dimitrios Pantazis[3], James J Levitt[4], Martha E Shenton[4], Guido Gerig[1,2]

Styner et al., 2006

Three problems of spherical harmonic representation

- Gibbs phenomenon (ringing artifacts)
- Computational bottleneck of solving large linear equations
- Slow convergence → Inefficient representation

(MICCAI 2008 workshop on mathematical foundations of computational anatomy)

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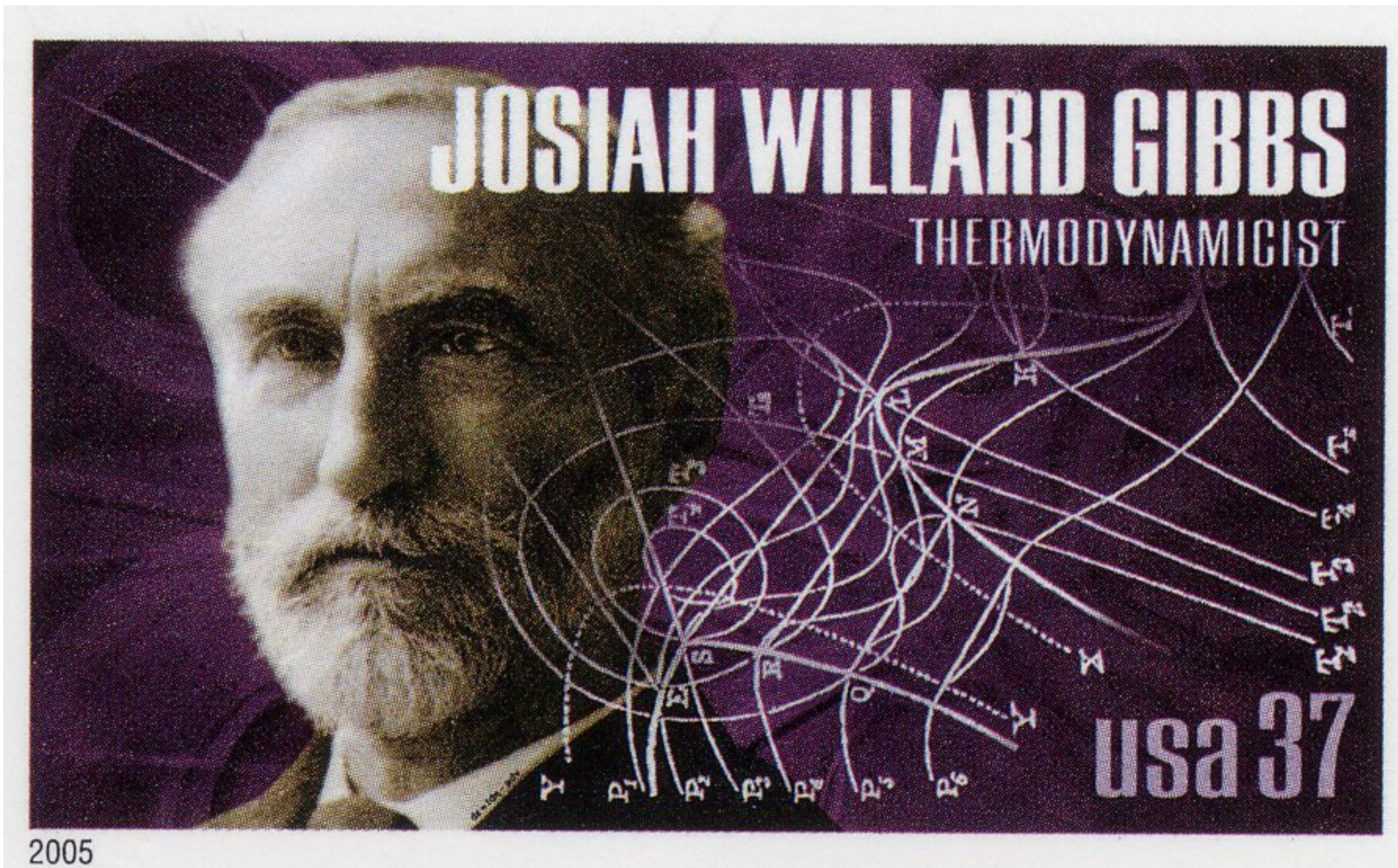
Weighted Fourier Analysis

Another reason why Fourier descriptors or SPHARM is not optimal

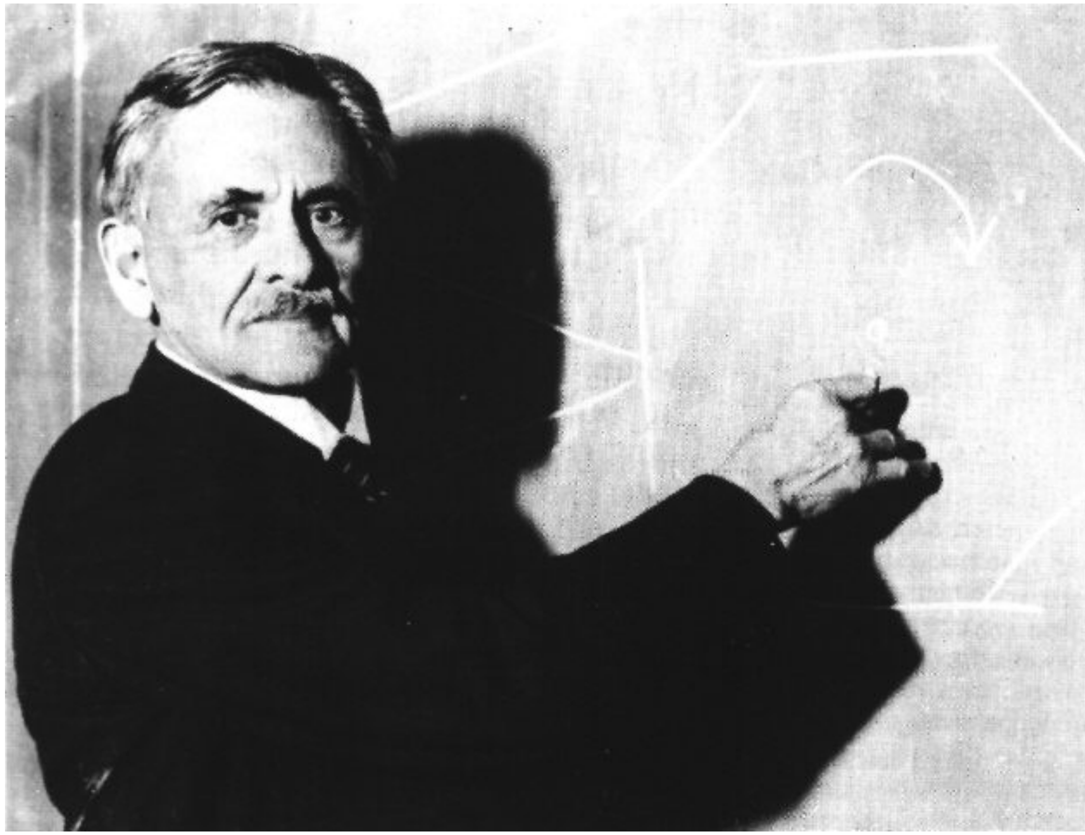
Gibbs phenomenon (ringing artifacts)



History of Gibbs Phenomenon

Mathematician Henry Willbraham published a paper on this in 1848 but did not attract any attention.

Albert Abraham Michelson

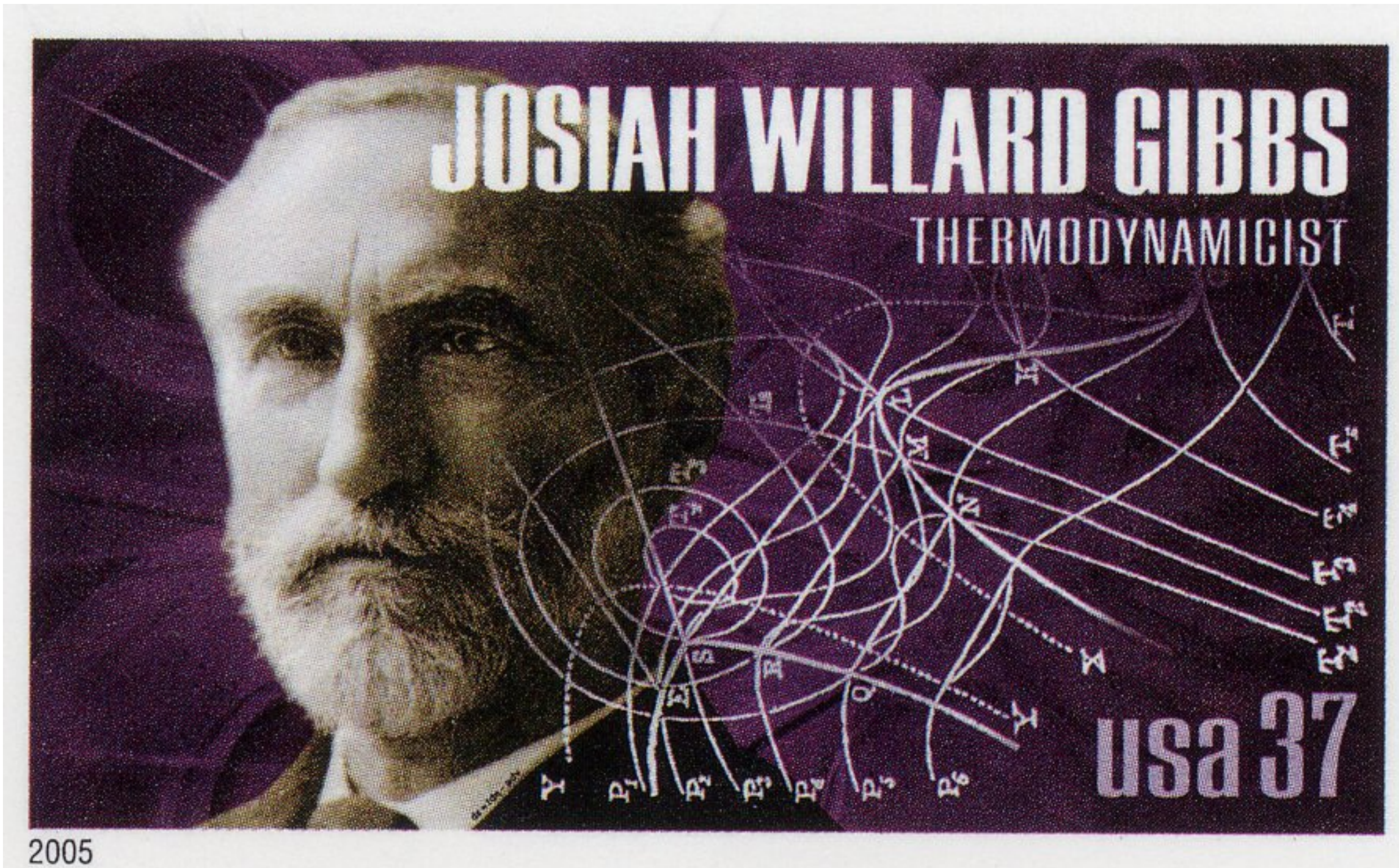


Observed the phenomenon but assumed it to be mechanical error

Harmonic Analyzer

The Michelson-Stratton harmonic analyzer, one of the first mechanical analogue computers, recorded data from spectroscopic experiments.

Josiah Willard Gibbs

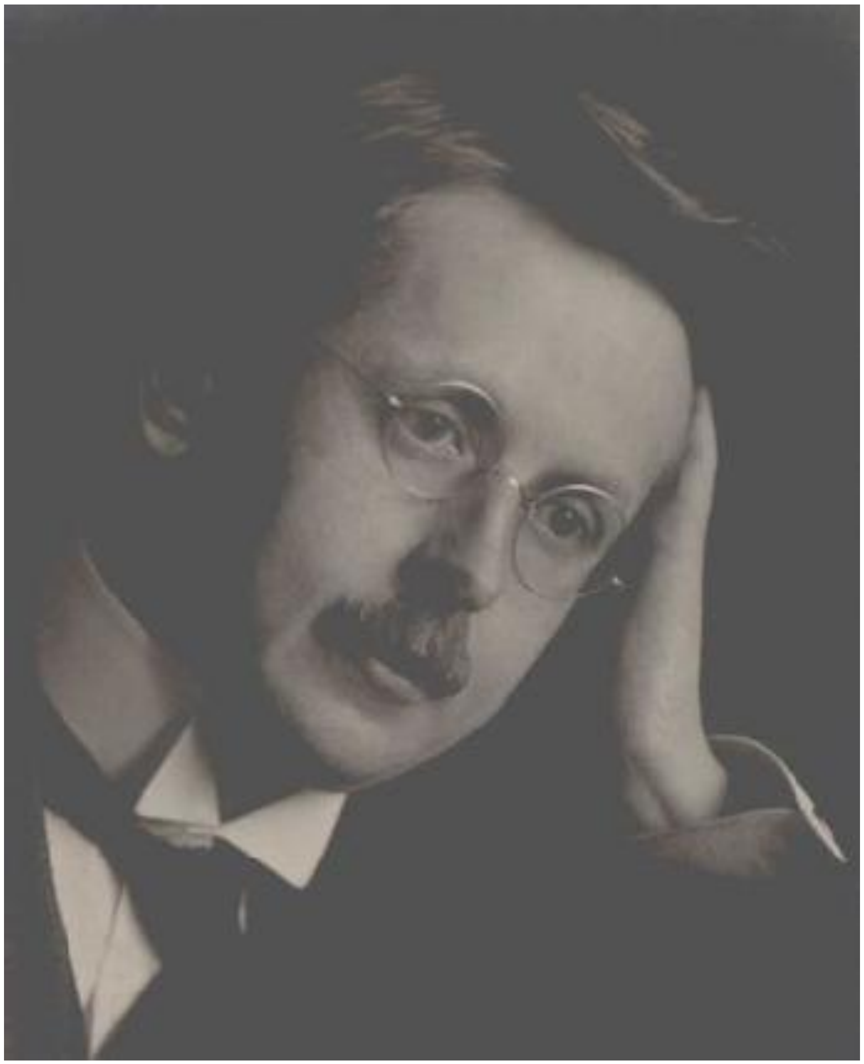


correctly explained the phenomenon as mathematical in Nature 1899.

Maxime Bocher

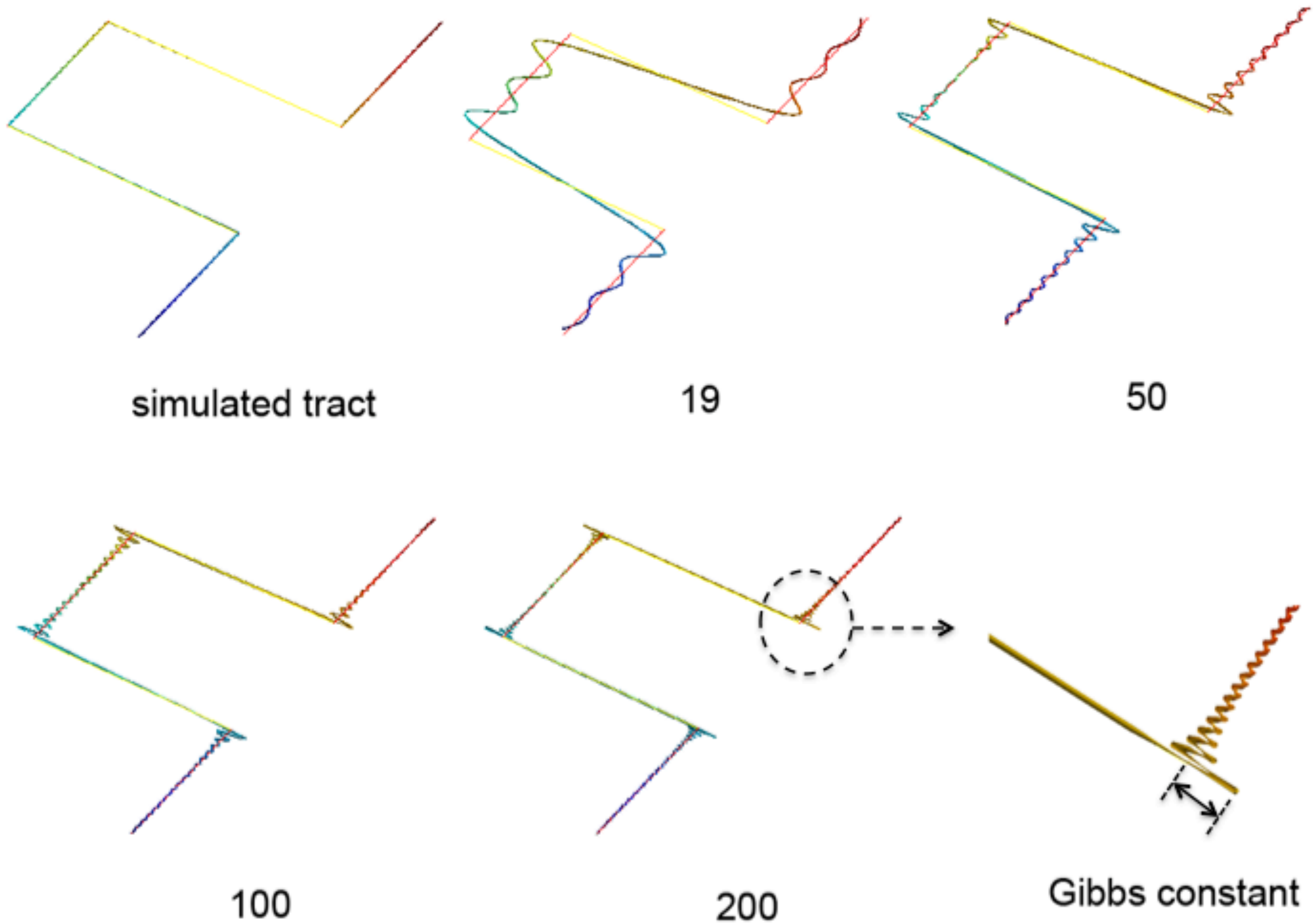
Gave detailed mathematical analysis and named it Gibbs phenomenon in 1906.

Herman Weyl

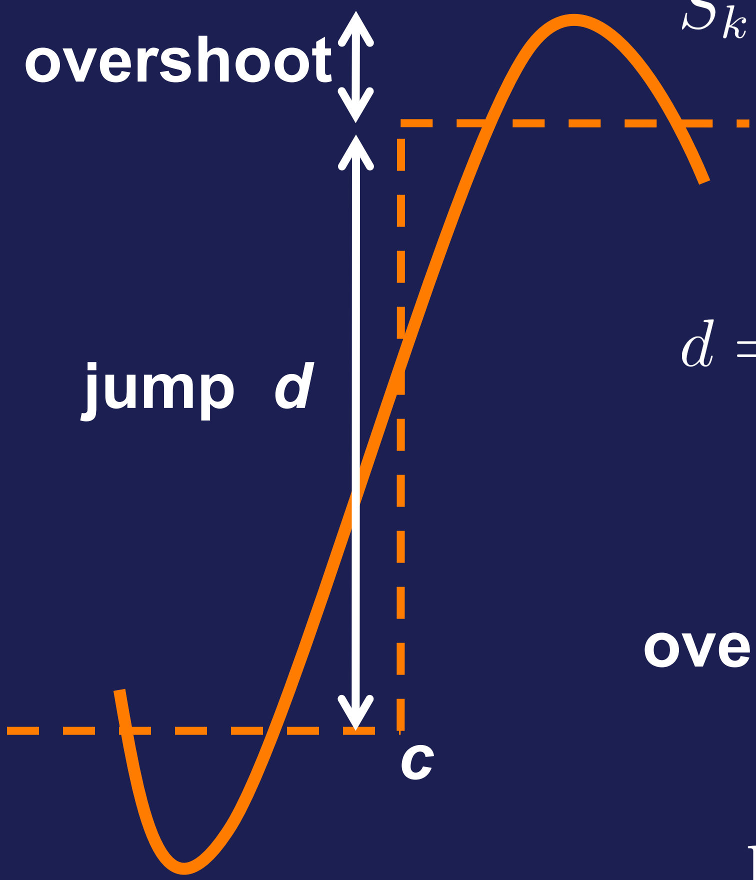


Investigated the Gibbs phenomenon associated with spherical harmonics in 1968.

Gibbs phenomenon on 3D curve
Overshooting at jump discontinuity



Limit of overshoot



$$S_k(u) = \sum_{j=0}^{k} f_j \psi(u)$$

$$d = \lim_{u \to c^+} f(u) - \lim_{u \to c^-} f(u) > 0$$

overshoot $S_k(u_\circ) - \lim_{u \to c^+} f(u)$

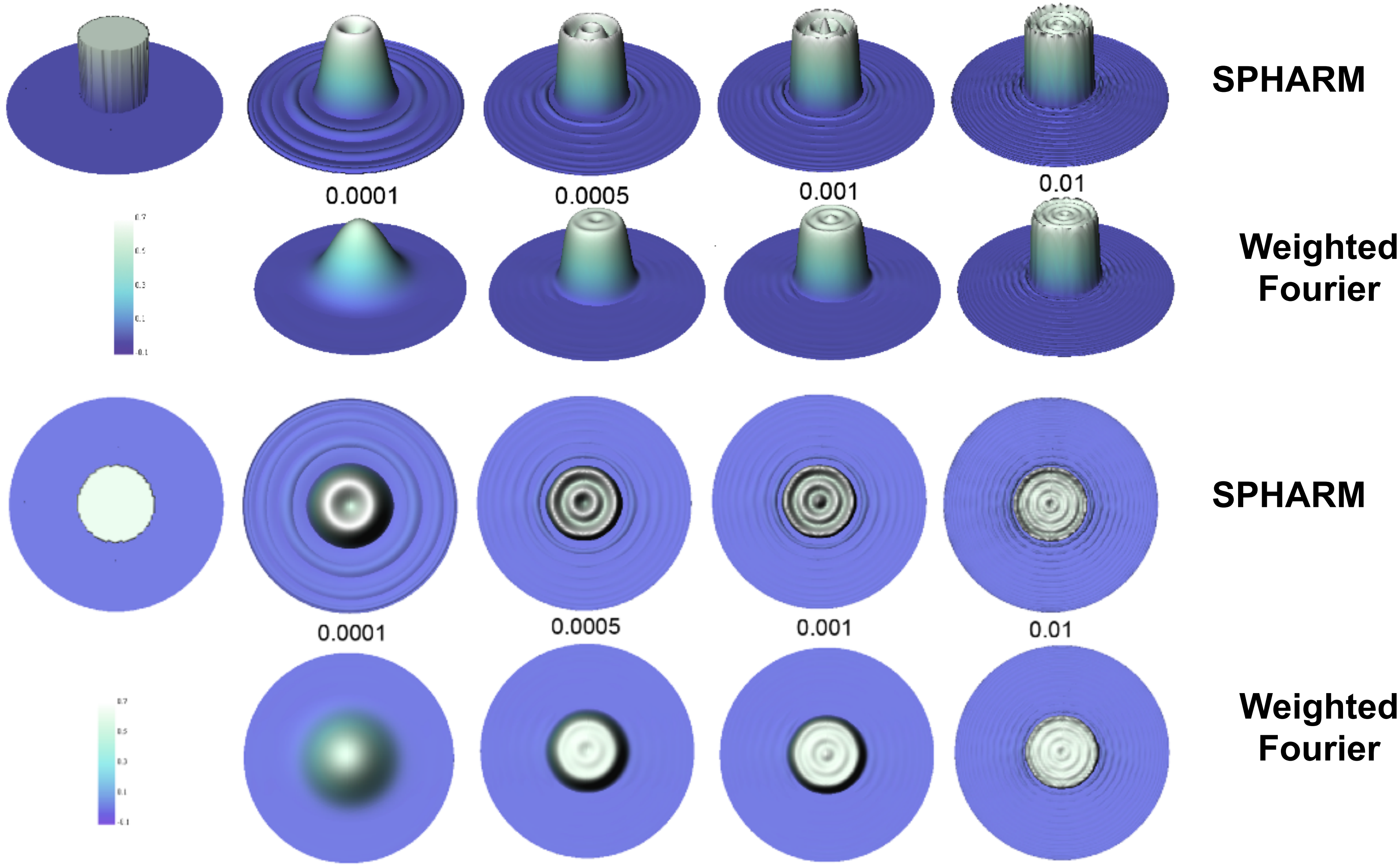
$$\lim_{k \to \infty} S_k(u_\circ) - \lim_{u \to c^+} f(u) = \frac{d}{2}(g - 1)$$

Gibbs constant

$$g = \frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\, dx = 1.17897974\cdots$$

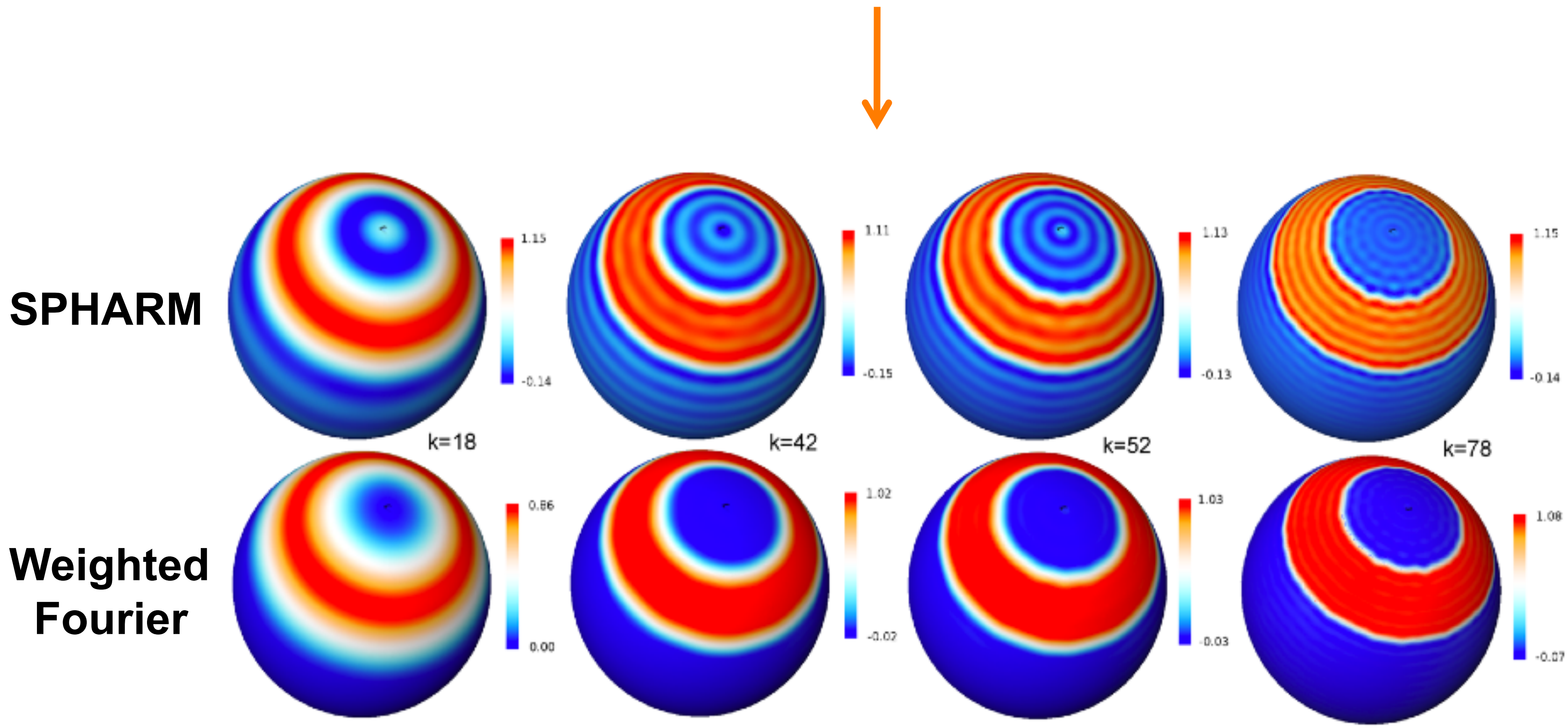
Why do we need to reduce it?

What is wrong with Previous approach
Gibbs phenomenon on shape

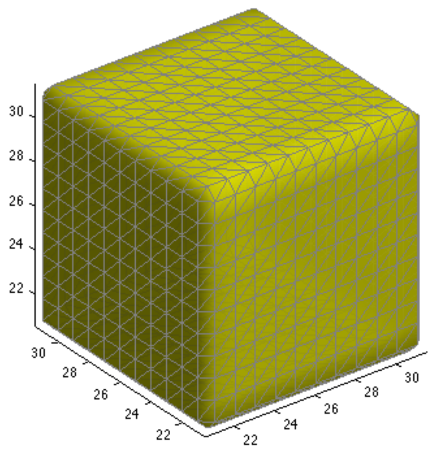


Reduction of Gibbs phenomenon (ringing artifacts)

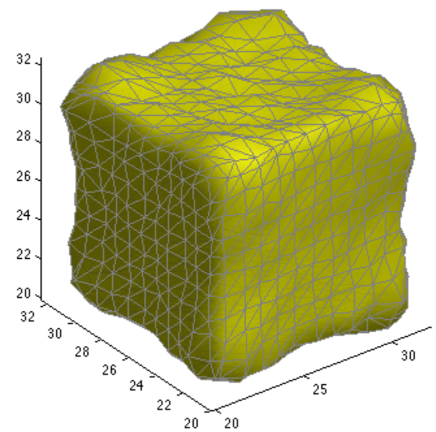
Initial signal = 1 in circular band
0 outside



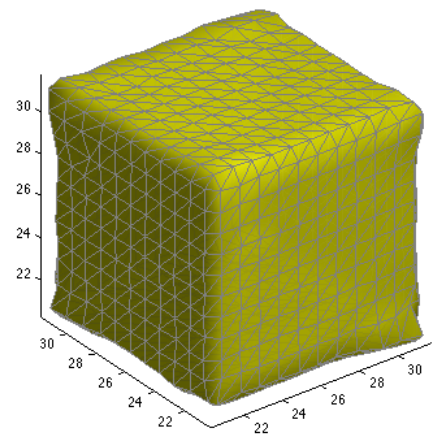
Gibbs phenomenon on a closed surface



Cube

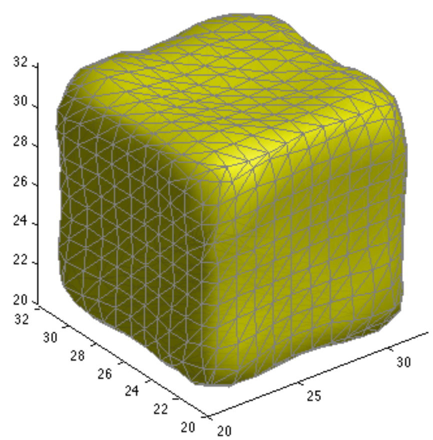


k=42 $\sigma=0$

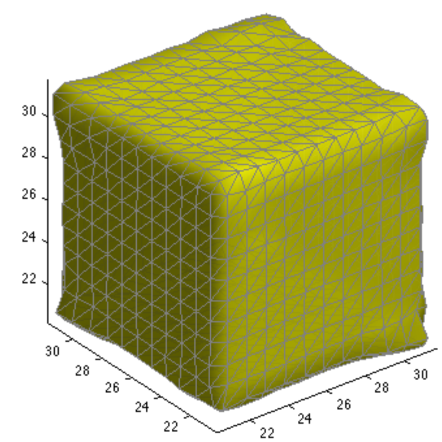


k=78 $\sigma=0$

Spherical harmonic representation



k=42 $\sigma = 0.001$

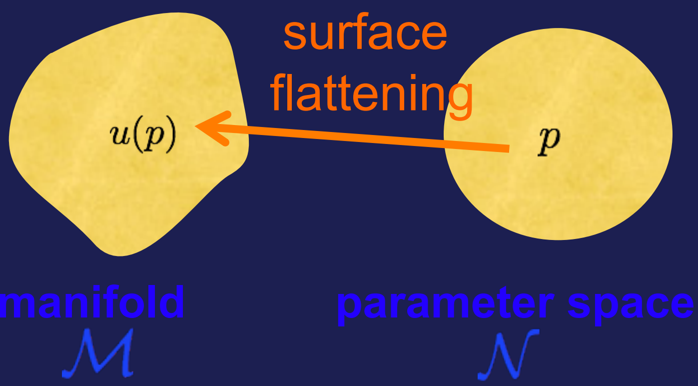


k=78 $\sigma = 0.0001$

Weighted Fourier

Chung et al., IEEE TMI, 2007

Cortical manifold and function defined on the manifold



Anatomical manifold $\mathcal{M} \in \mathbb{R}^d$

Parameter space $\mathcal{N} \in \mathbb{R}^m$

Hilbert space $L^2(\mathcal{N})$ with inner product

$$\langle g_1, g_2 \rangle = \int_{\mathcal{N}} g_1(p) g_2(p) \mu(p)$$

Self-adjoint operator $\mathcal{L}$

$$\langle \mathcal{L} g_1, g_2 \rangle = \langle g_1, \mathcal{L} g_2 \rangle$$

→

Basis function

$$\mathcal{L} \psi_j = \lambda_j \psi_j$$

Weighted Fourier Series

function defined on
surface +
surface coordinates

t = scale
bandwidth
diffusion time

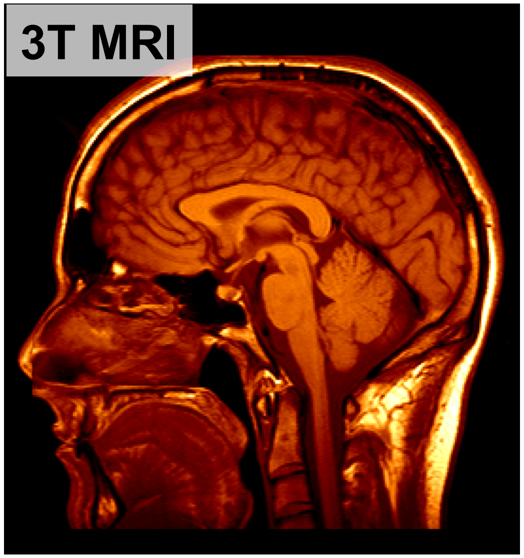
Self-adjoint PDE: $$\partial_t g + \mathcal{L} g = 0, g(p, t = 0) = f(p)$$

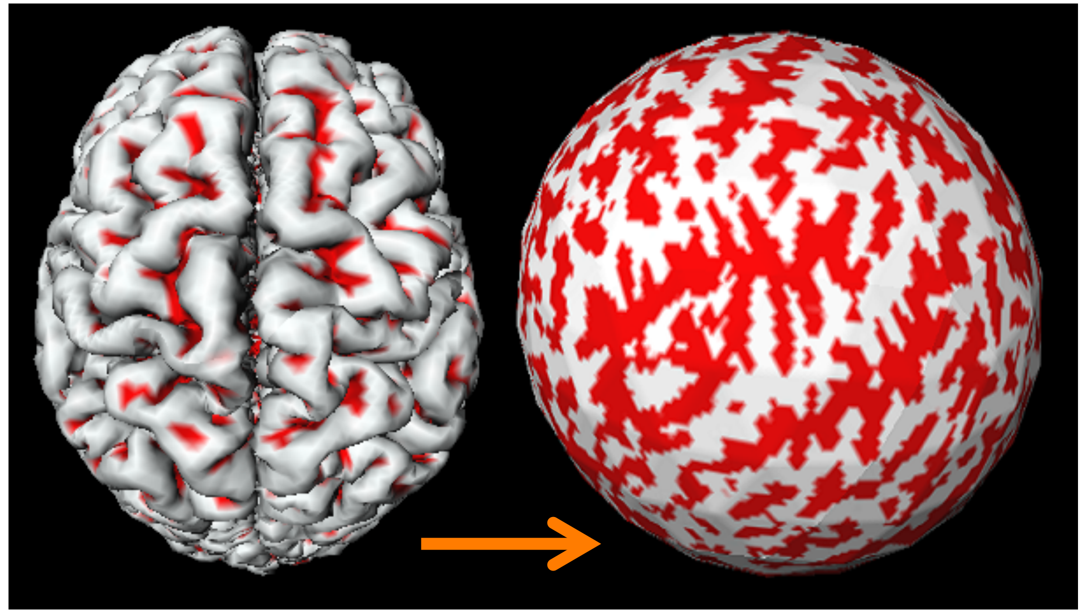
Analytic solution

Weighted Fourier Series $$g(p,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \langle f, \psi_j \rangle \psi_j(p)$$

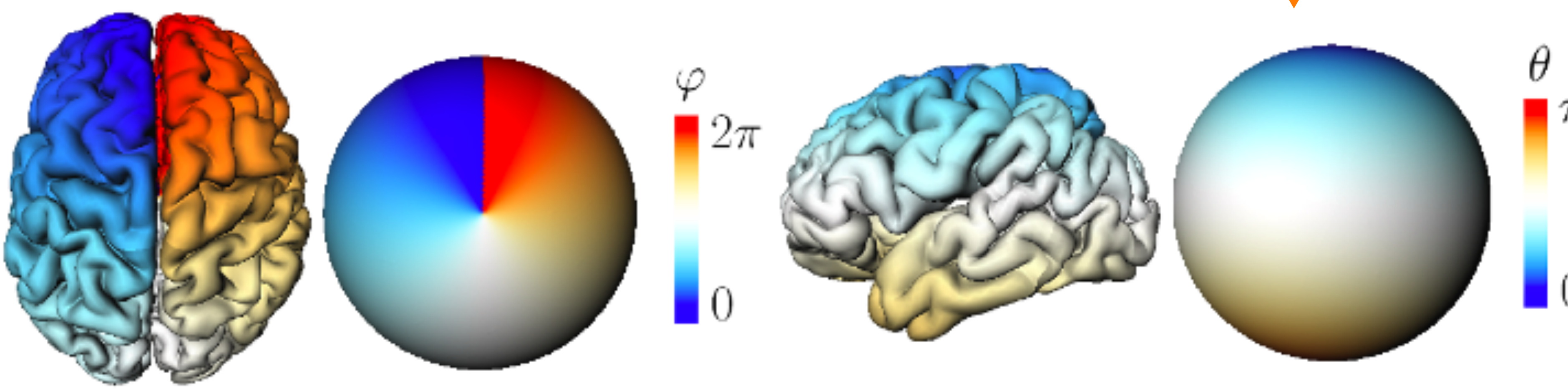
Isotropic Kernel Smoothing $$= \int_{\mathcal{N}} K_t(p,q) f(q) \, d\mu(q)$$

Cortical Surface Modeling



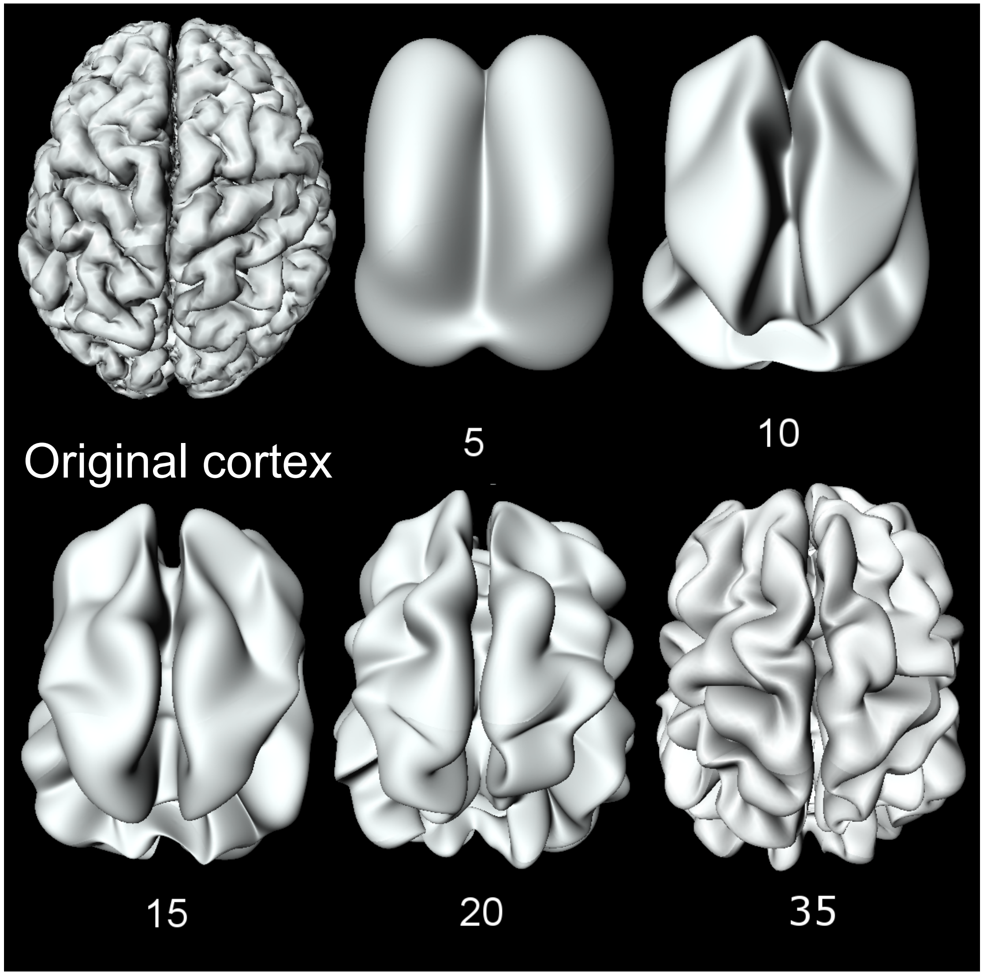


Deformable surface algorithm



Spherical angle based coordinate system

SPHARM representation up to degree 35



- Direct numerical integration takes more than 24 hours of computation.

- A faster implementation using the *iterative residual fitting algorithm (IRF)* that does computation in less than 10min.

SPHRM representation

•Given functional measurement *f(p)* on a unit sphere, it is modeled as

$$f(p) = \sum_{l=0}^{k} \sum_{m=-l}^{l} f_{lm} Y_{lm}(p) + e(p)$$

e: noise (image processing, numerical, biological)

f_{lm}: unknown Fourier coefficients

•The parameters are estimated in the least squares fashion.

- For measurements $f(p_1), f(p_2), \cdots, f(p_n)$, $(n > 46,000)$, we set up normal equations:

$$f(p_i) = \sum_{l=0}^{k} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}(p_i).$$

i-th mesh vertex

- Matrix form:

$$\underbrace{\begin{pmatrix} f(p_1) \\ f(p_2) \\ \vdots \\ f(p_n) \end{pmatrix}}_{\mathbf{F}} = \underbrace{\begin{pmatrix} Y_{00}(p_1) & Y_{1-1}(p_1) & Y_{10}(p_1) & \cdots & Y_{kk}(p_1) \\ Y_{00}(p_2) & Y_{1-1}(p_2) & Y_{10}(p_2) & \cdots & Y_{kk}(p_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{00}(p_n) & Y_{1-1}(p_n) & Y_{10}(p_n) & \cdots & Y_{kk}(p_n) \end{pmatrix}}_{\mathbf{Y}} \underbrace{\begin{pmatrix} \beta_{00} \\ \beta_{1-1} \\ \vdots \\ \beta_{kk} \end{pmatrix}}_{\beta}$$

40962 x 7000

Estimation: $\widehat{\beta} = (\mathbf{Y}'\mathbf{Y})^{-1}\mathbf{Y}'\mathbf{F}$.

Need to invert 7000 x 7000

Inverting large matrices

Direct computation:

Gauss-Jordan elimination → running time O(k^3)

LU-decomposition

QR-decomposition

Approximate iterative procedures:

Recursive Least squares estimation (RLSE)

Iterative residual fitting (IRF)

Gauss-Seidel

→ running time O(k^2)

How do we estimate spherical harmonic coefficients numerically?

Weighted-SPHARM

$$v_i(\theta,\varphi) = \sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} f_{lm}^{i} Y_{lm}(\theta,\varphi)$$

3 x 5000=15,000 Fourier coefficients

Iterative residual fitting (IRF) algorithm.

See Chung et al. TMI (2007) for detail.

Estimating 15,000 Fourier coefficients

Direct numerical integration takes forever.

Fast Fourier transform (FFT) is not fast either.

Iterative residual fitting (IRF) algorithm

1. Estimate the Fourier coefficients iteratively from lower degree to higher degree.
2. Break one huge linear problem (3GB) into many smaller linear problems (500MB).
3. At each iteration, residual is used to estimate the coefficients of next degree.

MATLAB implementation of

Iterative residual fitting (IRF) algorithm

MATLAB implementation can be downloaded from
http://www.stat.wisc.edu/~mchung/softwares/weighted-SPHARM/weighted-SPHARM.html

Sample cortical surface data is also provided.

Iterative residual fitting algorithm

Related to the matching pursuit method

IEEE TRANSACTIONS ON SIGNAL PROCESSING, VOL. 41, NO. 12, DECEMBER 1993 3397

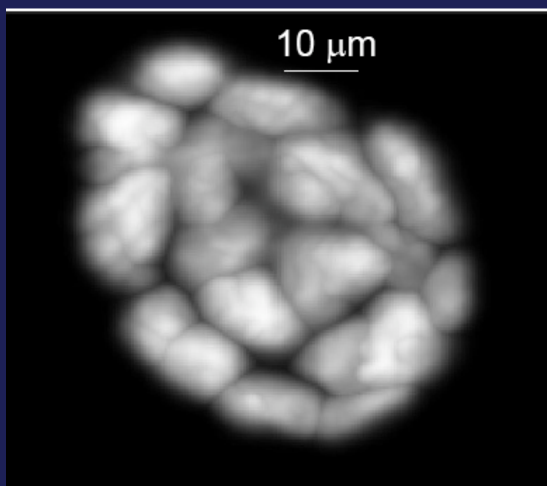
Matching Pursuits With Time-Frequency Dictionaries

Stéphane G. Mallat, *Member, IEEE*, and Zhifeng Zhang

Mallet and Zhang, 1993. IEEE Trans. Signal Processing

Application of iterative residual fitting algorithm

Reconstruction of 3D volume data using spherical harmonics



Multiple shells

Humongous linear system involving spherical harmonics

$$f(\theta, \varphi, r) = \sum_{i} \sum_{j} \sum_{k} \beta_{ijk} Y_{ijk}(\theta, \varphi, r)$$

Khairy et al., 2008 MICCAI

Iterative residual fitting (IRF) algorithm

Scalable approach to solving a huge linear equation

Step 1. measurements $f(p_1), \cdots, f(p_n)$

Step 2. Set initial degree=0 $k = 0$

Step 3. Solve $$f(p_i) = \sum_{m=-k}^{k} \beta_{km} Y_{km}(p_i)$$

Project data into a finite subspace

Step 3.5. $f \leftarrow f - \hat{f}$

Once low frequency parts are estimated, we throw them away

Step 4. Set degree $k \leftarrow k + 1$

Iterate

Determining the optimal degree via stepwise forward model selection framework

Consider the following $(k-1)$-th degree model

$$f(p_i) = \sum_{l=0}^{k-1} \sum_{m=-l}^{l} e^{-\lambda(l+1)\sigma} f_{lm} Y_{lm}(p_i) + \epsilon(p_i), \; i = 1, \cdots, n$$

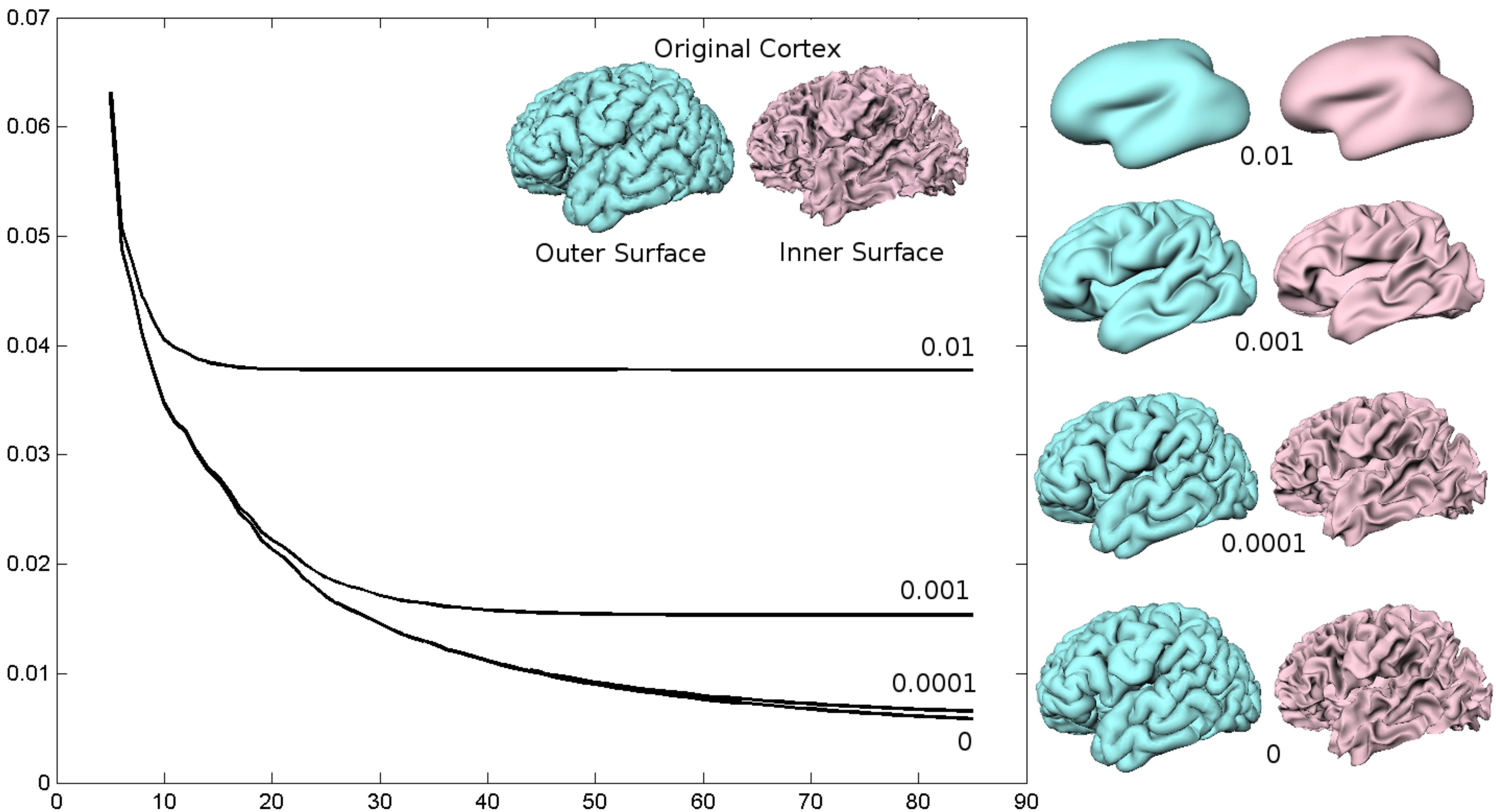
where ϵ are Gaussian random variables. Testing if the k-th degree model is better than the previous $(k-1)$-th degree model can be done by testing

$$H_0 : f_{km} = 0 \text{ for all } -k \leq m \leq k.$$

Then under the null hypothesis, the test statistic is

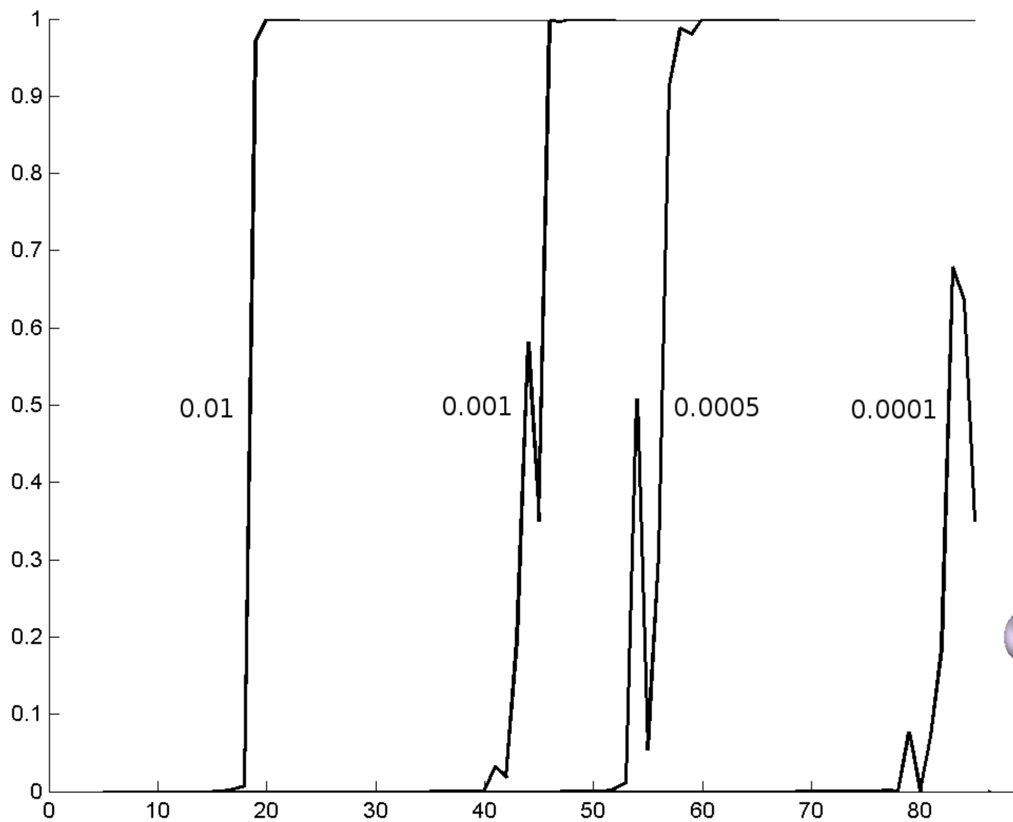
$$F = \frac{(\text{SSE}_{k-1} - \text{SSE}_k)/(2k+1)}{\text{SSE}_{k-1}/(n-(k+1)^2)} \sim F_{2k+1, n-(k+1)^2}$$

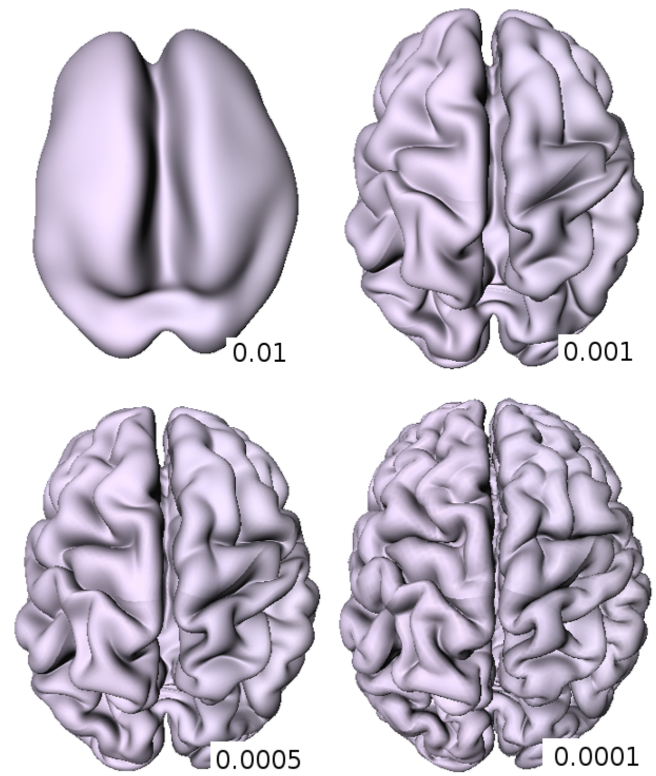
Weighted-SPHARM at the 80th degree for different bandwidth



Root mean squared error (RMSE)
= error between original surface and weighted-SPHARM

For each bandwidth σ , optimal degree is automatically selected via **forward best model selection procedure**.

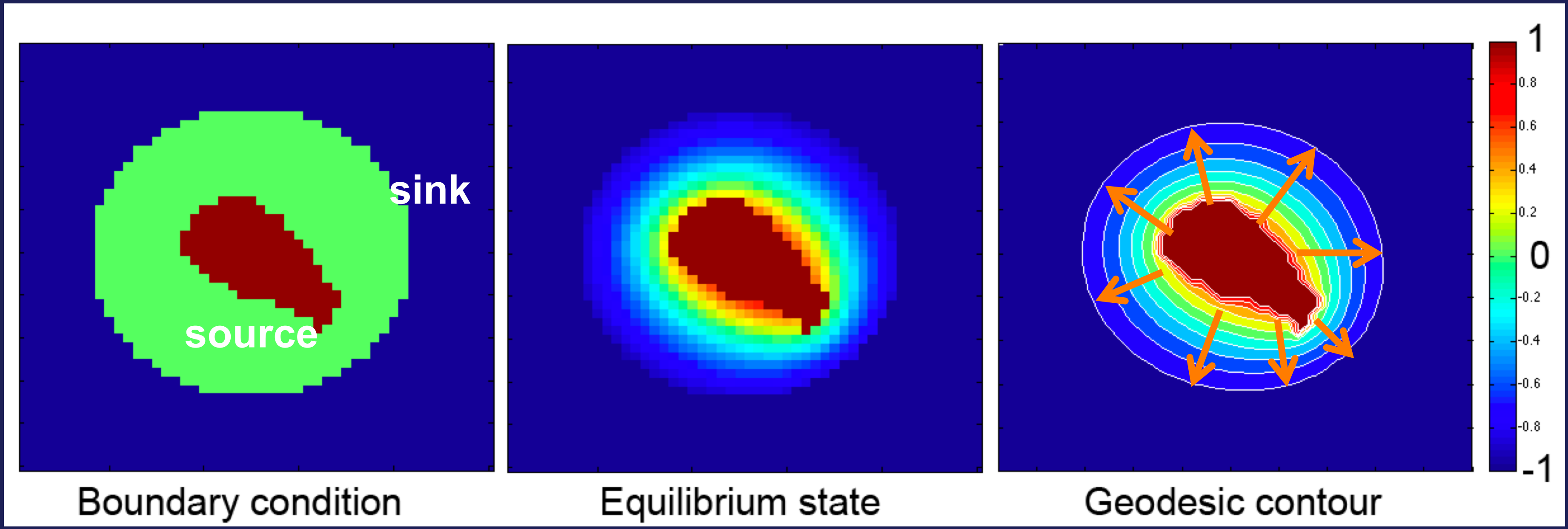




Optimal degree= first P-value >0.05

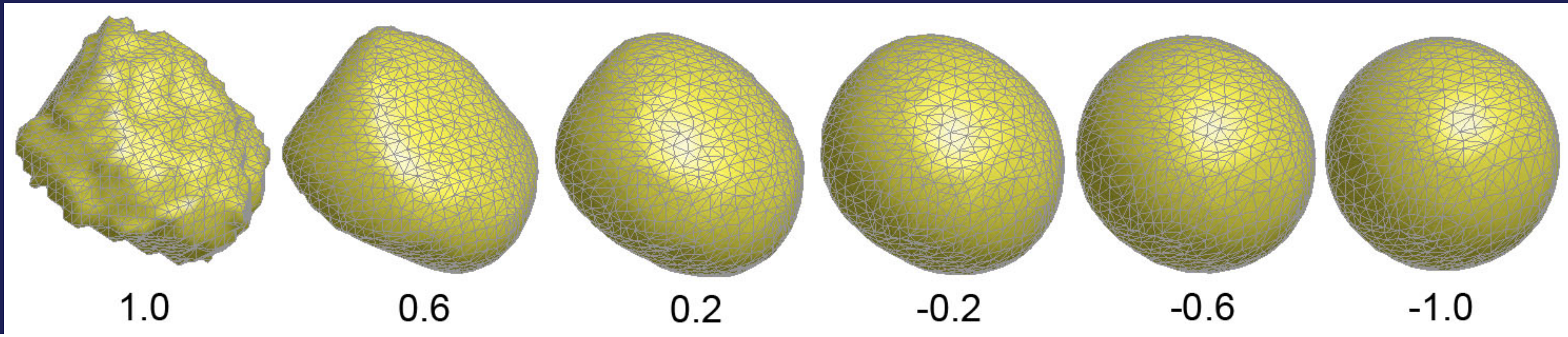
MATLAB Demonstration

Surface flattening (mapping to a parameter space)



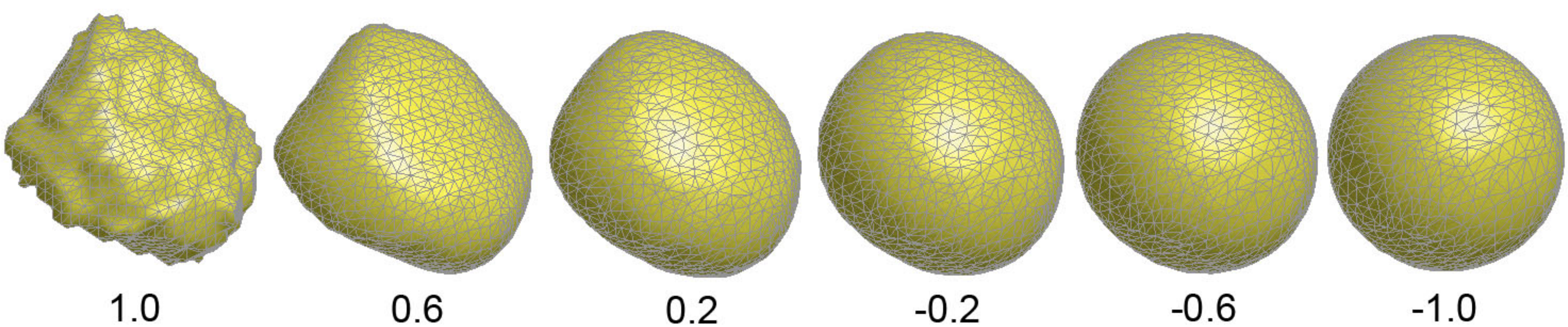
$$\frac{\partial f}{\partial \sigma} = \Delta f \xrightarrow{\text{Equilibrium state}} \Delta f = 0$$

$$f(\mathtt{source}, \sigma) = 1, f(\mathtt{sink}, \sigma) = -1$$

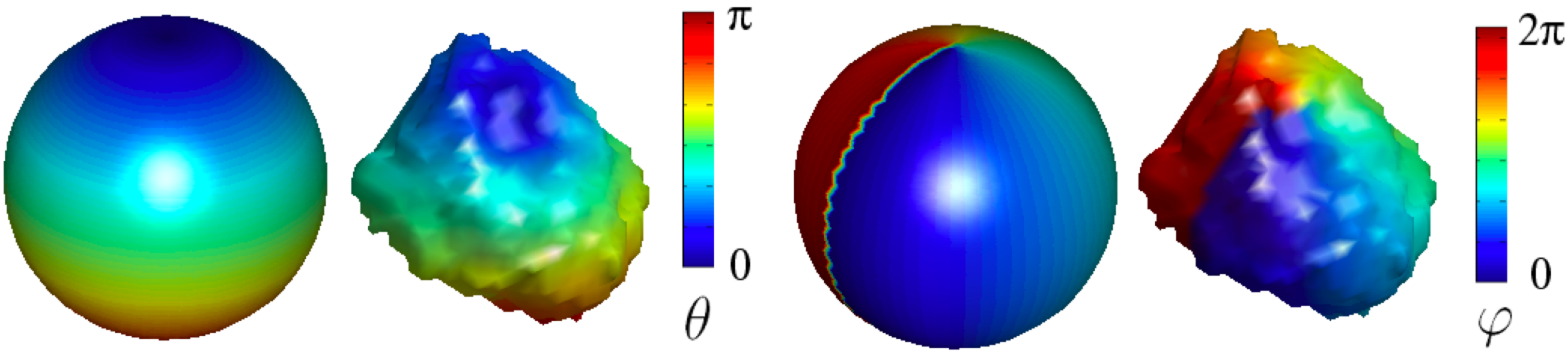


Follow the trajectory of heat diffusion

Tracing a path normal to contours, we obtain a unique smooth map from the amygdala to a sphere.



Euler angle based coordinate system for amygdala surface



Resolution of coordinate system:
about 1500 mesh vertices per amygdala

MATLAB Demonstration

Lecture 5

Parametric modeling of curvilinear structures (white matter fiber tracts, corpus callosum boundary, sulcal pattern)

Fourier descriptors

Cosine series representation

Read
chung.2010.SII
wang.2005.TR1113