Computational Methods in NeuroImage Analysis

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Lecture 3
Spherical Harmonic Representation (SPHARM)

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Spherical harmonic (SPHARM) representation

It is a technique for parameterizing anatomical boundaries using the spherical harmonic basis.

The surface coordinates x, y, z are expressed as a linear combination of basis functions. For instance,

\[ x(p) = \sum_{j=0}^{k} \beta_j \psi_j(p) \]
Parameterization using polynomials

\[ x(p) = \sum_{j=0}^{k} \beta_j \psi_j(p) \]

We use \( \{1, p, p^2, p^3, p^4, \ldots\} \) as a basis.

Parameters are estimated using the least squares method.
Estimating Fourier coefficients

• For each point \( p_i \), we have measurement \( f(p_i) \).

• Corresponding Fourier series:

\[
 f(p_i) = \beta_0 \phi_0(p_i) + \beta_0 \phi_0(p_i) + \cdots + \beta_k \phi_k(p_i)
\]

• Matrix form:

\[
 F = \Phi \beta
\]

\[
 \beta = (\Phi' \Phi)^{-1} \Phi' F
\]

• This is a nontrivial linear problem

See MATLAB demonstration
CMN.lecture03.SPHARM.09.17.2010.m
Surface Parameterization via quadratic surface

Global: tensor splines, SPHARM
Local: quadratic surface fitting

\[
X(u^1, u^2) = \begin{pmatrix}
x_1(u^1, u^2) \\
x_2(u^1, u^2) \\
x_3(u^1, u^2)
\end{pmatrix}
\]

\[
s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots
\]
Motivation for surface parameterization

Compared to other 3D volumetric techniques, surface based approach can quantify cortical variations better.

Ventricle enlargement

Age 14

Age 19
Final surface extraction result

Inner surface

Outer surface
3T MRI

- Tissue segmentation
- Surface extraction
- Triangle mesh with 1 million triangles

Continuous parameterization by spherical harmonics

Yellow: outer cortical surface
Blue: inner cortical surface
## Data structure for polygonal mesh (autism_cortical_surface.mat)

### Coordinates for subject 1

<table>
<thead>
<tr>
<th>Vertex</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>....</th>
<th>40962</th>
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<td>3.0</td>
<td>2.1</td>
<td>3.4</td>
<td>4.5</td>
<td></td>
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</tbody>
</table>

### Coordinates for subject 2

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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>....</th>
<th>40962</th>
</tr>
</thead>
<tbody>
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<td></td>
</tr>
<tr>
<td>Thickness</td>
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<td>3.4</td>
<td>2.7</td>
<td>5.1</td>
<td>3.7</td>
<td>4.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Corresponding vertices have approximate anatomical homology.
Quadratic surface fitting

\[ s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2 \beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots \]

Riemannian metric tensors

\[ g = \begin{pmatrix} 1 + \beta_1^2 & \beta_1 \beta_2 \\ \beta_1 \beta_2 & 1 + \beta_2^2 \end{pmatrix} \quad l = \begin{pmatrix} \beta_3 & \beta_4 \\ \beta_4 & \beta_5 \end{pmatrix} \]

Mean curvature

\[ K_M = \frac{\text{tr}(g^{-1}l)}{2} = \frac{\beta_3(1 + \beta_2^2) + \beta_5(1 + \beta_1^2) - 2\beta_1 \beta_2 \beta_4}{2 + 4(\beta_1^2 + \beta_2^2)} \]
Polynomial Regression on irregular triangular mesh

\[ Y = X\beta \]

\[
\begin{pmatrix}
  u_1^3 \\
  u_2^3 \\
  \vdots \\
  u_m^3
\end{pmatrix}
= 
\begin{pmatrix}
  u_1^1 & u_1^2 & (u_1^1)^2 & u_1^1 u_1^2 & (u_1^2)^2 \\
  u_2^1 & u_2^2 & (u_2^1)^2 & u_2^1 u_2^2 & (u_2^2)^2 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  u_m^1 & u_m^2 & (u_m^1)^2 & u_m^1 u_m^2 & (u_m^2)^2
\end{pmatrix}
\begin{pmatrix}
  \beta_1 \\
  \beta_2 \\
  \vdots \\
  \beta_m
\end{pmatrix}
\]

\[ \hat{\beta} = (X'X)^{-1}X'Y \]
Bending energy or thin-plate spline energy can be used to measure the curvature of the surface. Between ages 12 and 16, it increases both locally and globally.
Principle curvature maps projected on the average template
Curvature change $t$ map between age 12 and 16
Compute cortical curvature and map curvature to unit sphere

3D problem

2D problem

Unit sphere gives a natural coordinate system (spherical coordinates).
Sulcal pattern matching by minimizing objective function = curvature difference - smoothness of deformation

See Paul Thompson’s earlier IEEE TMI paper
Surface area expansion/shrinking

Local surface area element:

$$\sqrt{|g|} = \sqrt{1 + \beta_1^2 + \beta_2^2}$$

Spherical harmonic representation was used to analytically compute and smooth surface area element.
Local area expansion with respect to a template (it ranges between 0 and 1.3)
Surface area change $t$ map

dilatation rate between age 12 and 16

min = -57 %  mean = -0.02 %  max = 65 %
Laplace-Beltrami Operator

\[ \Delta_X F = \frac{1}{|g|^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u^i} \left( |g|^{1/2} g^{ij} \frac{\partial F}{\partial u^j} \right) \]

Estimating differential operator on manifolds

\[ \tilde{\Delta} F(p_0) = w_0 F(p_0) + w_1 F(p_1) + \cdots + w_m F(p_m) \]
Estimation via conformal transformation

\[ s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots \]

\[ g = \begin{pmatrix} 1 + \beta_1^2 & \beta_1 \beta_2 \\ \beta_1 \beta_2 & 1 + \beta_2^2 \end{pmatrix} \]

Laplace-Beltrami operator is invariant

\[ s(v^1, v^2) = \gamma_1 (v^1)^2 + \gamma_2 v^1 v^2 + \gamma_3 (v^2)^2 + \cdots \]

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ \Delta X = \frac{1}{\lambda} \left( \frac{\partial^2}{\partial^2 u^1} + \frac{\partial^2}{\partial^2 u^2} \right) \]
Estimating the planar Laplacian on irregular triangular mesh

\[ Y = X\beta \]
Thin Plate Spline Parameterization

Measurement $f$ is represented as

$$f(p) = \sum_i \alpha_i \phi_i(p) + \sum_j \beta_j \varphi(p - p_j)$$

where $\phi_i$ is polynomial basis and $\varphi$ is the TPS radial basis

Parameters are estimated by minimizing

$$\min_f \sum_{i=1} |y_i - f(p_i)|^2 + \lambda J_3^2(f),$$
Thin Plate Spline (TPS) segmentation and modeling

TPS represents anatomical boundary as the zero level set of smooth function consists of polynomial and radial basis functions (Wahba, 1990; Xie et al., 2005a).
Spherical Harmonic (SPHARM) Representation

- Spherical harmonics are basis functions on a unit sphere.
- SPHARM can be used to construct the Fourier series expansion of a functional measurement.
- SPAHRM has been used in parameterizing anatomical boundary.
- New more localized approaches: wavelets, weighted-SPAHRM.
Spherical harmonics

$Y_{lm}$ is called the *spherical harmonic* of degree $l$ and order $m$.

$$Y_{lm} = \begin{cases} 
  c_{lm} P_l^{|m|} (\cos \theta) \sin(|m| \varphi), & -l \leq m \leq -1, \\
  \frac{c_{lm}}{\sqrt{2}} P^0_l (\cos \theta), & m = 0, \\
  c_{lm} P_l^{|m|} (\cos \theta) \cos(|m| \varphi), & 1 \leq m \leq l,
\end{cases}$$

where $c_{lm} = \sqrt{\frac{2l+1}{2\pi}} \frac{(l-|m|)!}{(l+|m|)!}$ and $P_l^m$ is the associated Legendre polynomials of order $m$. 

Spherical harmonic of degree $l$ and order $m$

$$Y_{lm} = \begin{cases} 
  c_{lm} P_l^m (\cos \theta) \sin(|m|\varphi), & -l \leq m \leq -1, \\
  \frac{c_{lm}}{\sqrt{2}} P_l^0 (\cos \theta), & m = 0, \\
  c_{lm} P_l^m (\cos \theta) \cos(|m|\varphi), & 1 \leq m \leq l, 
\end{cases}$$
SPHRM representation

• Given functional measurement $f(p)$ on a unit sphere, it is modeled as

$$f(p) = \sum_{l=0}^{k} \sum_{m=-l}^{l} f_{lm} Y_{lm}(p) + e(p)$$

$e$: noise (image processing, numerical, biological)

$f_{lm}$: unknown Fourier coefficients

• The parameters are estimated in the least squares fashion.
• For measurements $f(p_1), f(p_2), \cdots, f(p_n)$, $(n > 46,000)$, we set up normal equations:

$$f(p_i) = \sum_{l=0}^{k} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}(p_i).$$

• Matrix form:

$$\begin{pmatrix} f(p_1) \\ f(p_2) \\ \vdots \\ f(p_n) \end{pmatrix} = \begin{pmatrix} Y_{00}(p_1) & Y_{-11}(p_1) & Y_{10}(p_1) & \cdots & Y_{kk}(p_1) \\ Y_{00}(p_2) & Y_{-11}(p_2) & Y_{10}(p_2) & \cdots & Y_{kk}(p_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{00}(p_n) & Y_{-11}(p_n) & Y_{10}(p_n) & \cdots & Y_{kk}(p_n) \end{pmatrix} \begin{pmatrix} \beta_{00} \\ \beta_{1-1} \\ \vdots \\ \beta_{kk} \end{pmatrix}.$$ 

Estimation: $\hat{\beta} = (Y'Y)^{-1}Y'F.$
Cortical Surface Modeling

3T MRI

Deformable surface algorithm

Spherical angle based coordinate system
Mapping from cortex to unit sphere
Each x, y, z Cartesian coordinates are represented independently.
Original Cortex

Outer Surface

Inner Surface

80 degree SPHARM
FreeSurfer results
Gibbs phenomenon (ringing artifacts) on surface

Severe distortion at low degree

Exponentially Weighted SPHARM

Cube

k=42  σ=0

k=78  σ=0

k=42  σ= 0.001

k=78  σ = 0.0001
Determining the optimal degree via stepwise forward model selection framework

Consider the following \((k - 1)\)-th degree model

\[
f(p_i) = \sum_{l=0}^{k-1} \sum_{m=-l}^{l} e^{-\lambda(l+1)\sigma} f_{lm} Y_{lm}(p_i) + \epsilon(p_i), \quad i = 1, \ldots, n
\]

where \(\epsilon\) are Gaussian random variables. Testing if the \(k\)-th degree model is better than the previous \((k - 1)\)-th degree model can be done by testing

\[
H_0 : f_{km} = 0 \text{ for all } -k \leq m \leq k.
\]

Then under the null hypothesis, the test statistic is

\[
F = \frac{(\text{SSE}_{k-1} - \text{SSE}_k) / (2k + 1)}{\text{SSE}_{k-1} / (n - (k + 1)^2)} \sim F_{2k+1, n-(k+1)^2}
\]
Weighted-SPHARM at the 80th degree for different bandwidth

Root mean squared error (RMSE) = error between original surface and weighted-SPHARM
For each bandwidth $\sigma$, optimal degree is automatically selected via **forward best model selection procedure**.

Optimal degree = first P-value >0.05
Weighted-SPHARM at different bandwidth

- The degree is selected automatically.
- The only free parameter in the model is the bandwidth.
SPHARM estimation of cortical thickness

Thickness estimation based on traditional method

Too much smoothing
Weighted-SPHARM of cortical thickness
Weighted-SPHARM at different scale

Original X-coordinate
78th degree SPHARM representation

The coefficients are treated as a multivariate measure and feed into classification techniques.
Statistical Model: Karhunen-Loeve expansion

Uncorrelated normal

Classification? Discrimination?

\[
\sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)t} \langle f, Y_{lm} \rangle Y_{lm}(\theta, \varphi)
\]
MATLAB Demonstration
Lecture 4

Iterative linear model fitting methods:

Matching Pursuits

Iterative Residual Fitting (IRF) algorithm

Read

chung.2008.sinica
mallat.1993