Computational Methods in NeuroImage Analysis

Instructor: Moo K. Chung
mkchung@wisc.edu

Lecture 3
Spherical Harmonic Representation (SPHARM)

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Read Chapter 7 of the textbook

Spherical harmonic (SPHARM) representation

It is a technique for parameterizing anatomical boundaries using the spherical harmonic basis.

The surface coordinates x,y, z are expressed as a linear combination of basis functions. For instance,

$$x(p) = \sum_{j=0}^{k} \beta_j \psi_j(p)$$

Parameterization using polynomials

$$x(p) = \sum_{j=0}^{k} \beta_j \psi_j(p)$$

We use {1, p, p^2, p^3, p^4, …} as a basis.

Parameters are estimated using the least squares method.

Estimating Fourier coefficients

- For each point p_i , we have measurement $f(p_i)$.
- Corresponding Fourier series:

$$f(p_i) = \beta_0\phi_0(p_i) + \beta_0\phi_0(p_i) + \cdots + \beta_k\phi_k(p_i)$$

- Matrix form:

$$F = \Phi\beta$$

$$\beta = (\Phi^{'}\Phi)^{-1}\Phi^{'}F$$

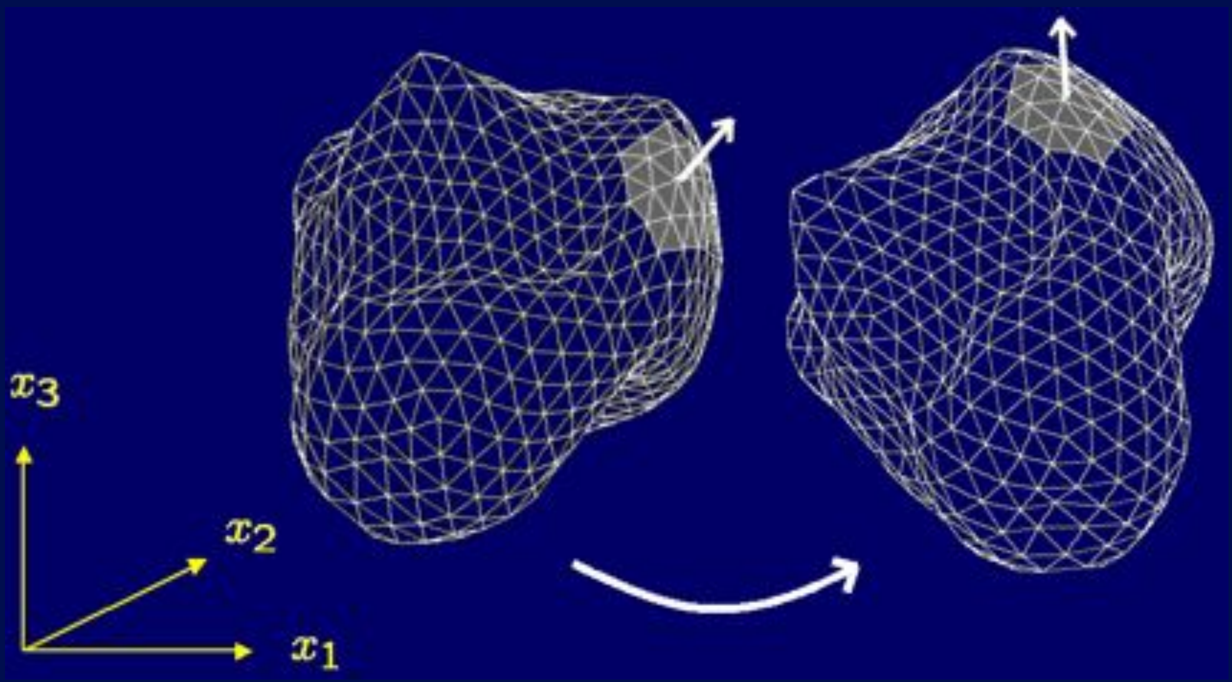
- This is a nontrivial linear problem

See MATLAB demonstration
CMN.lecture03.SPHARM.09.17.2010.m

Surface Parameterization via quadratic surface

Global: tensor splines, SPHARM
Local: quadratic surface fitting

$$X(u^1, u^2) = \begin{pmatrix} x_1(u^1, u^2) \\ x_2(u^1, u^2) \\ x_3(u^1, u^2) \end{pmatrix}$$



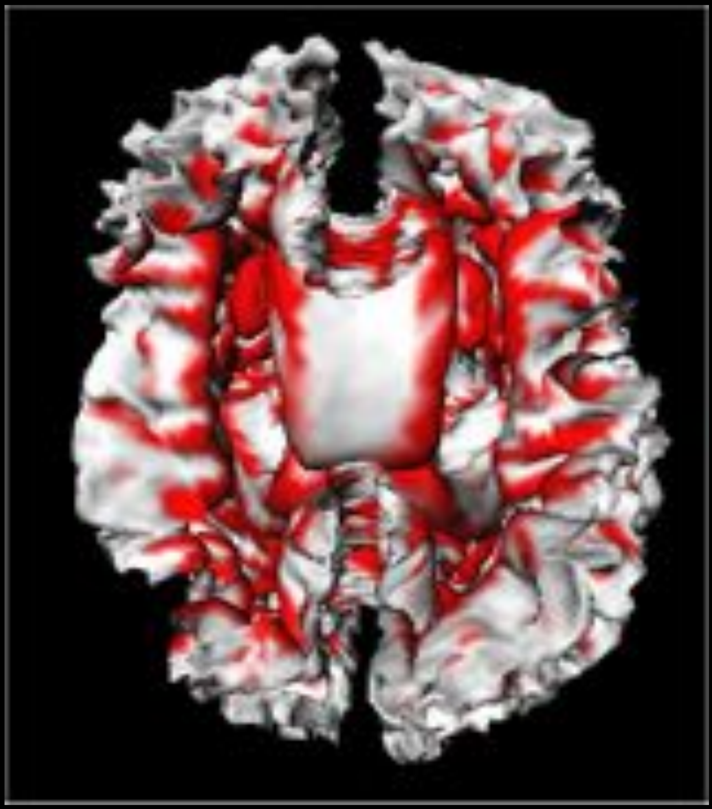
$$s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots$$

Motivation for surface parameterization

Compared to other 3D volumetric techniques, surface based approach can quantify cortical variations better.

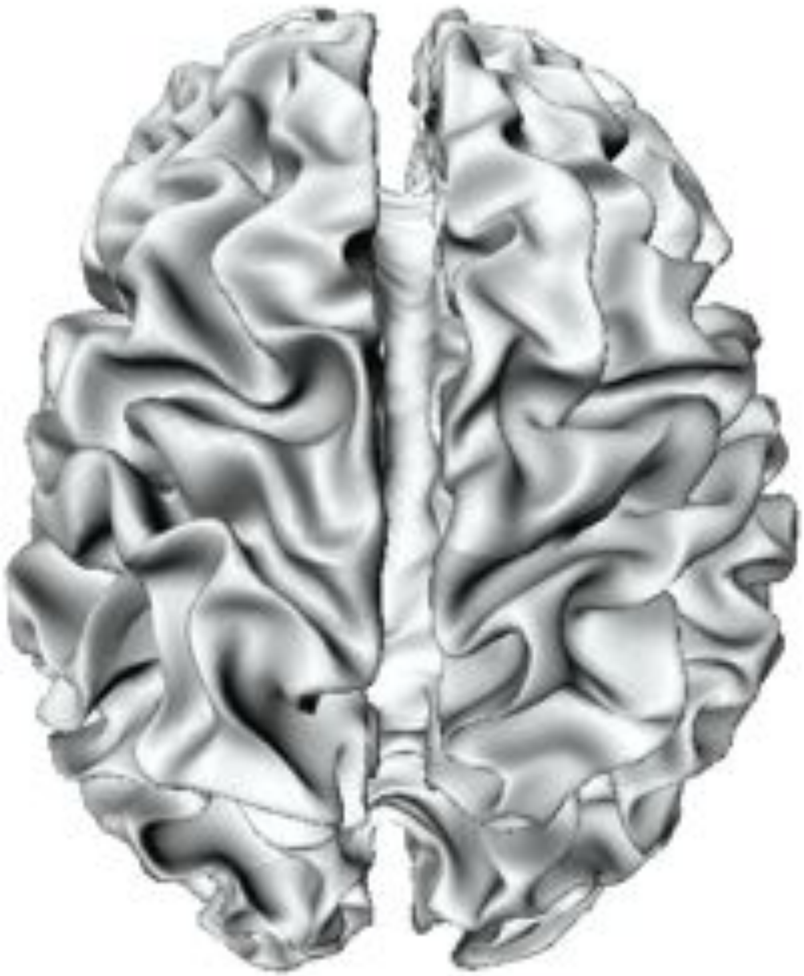
Age 14

Ventricle enlargement

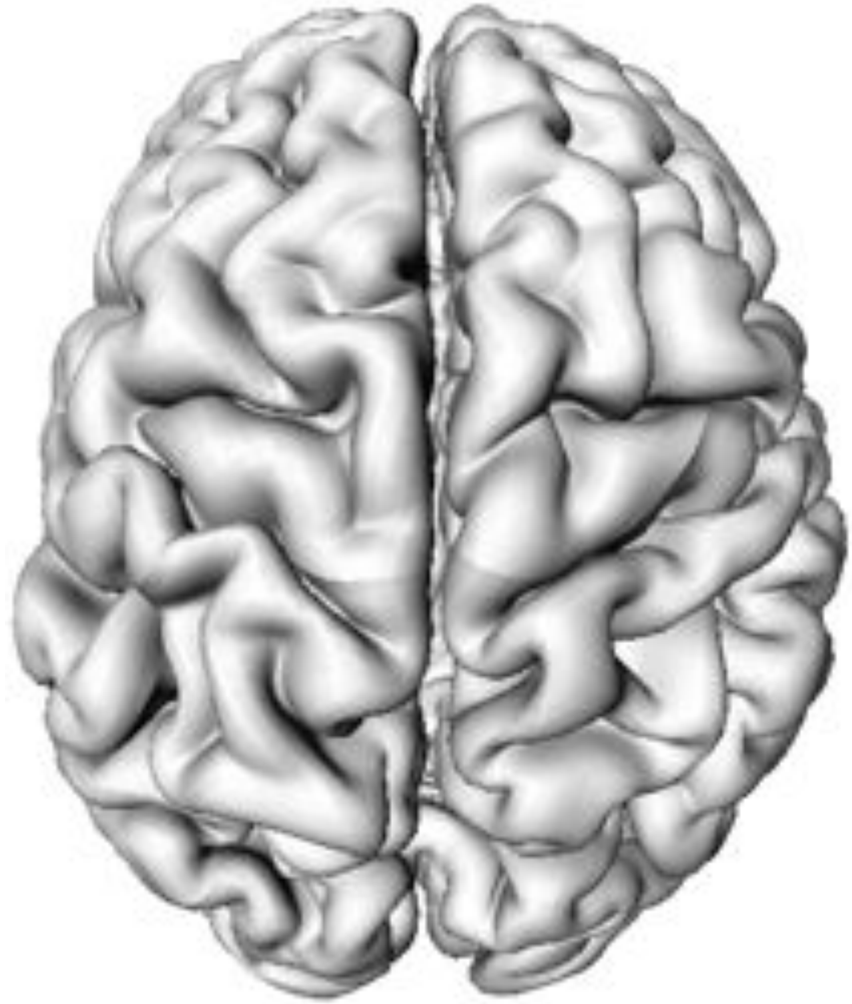


Age 19

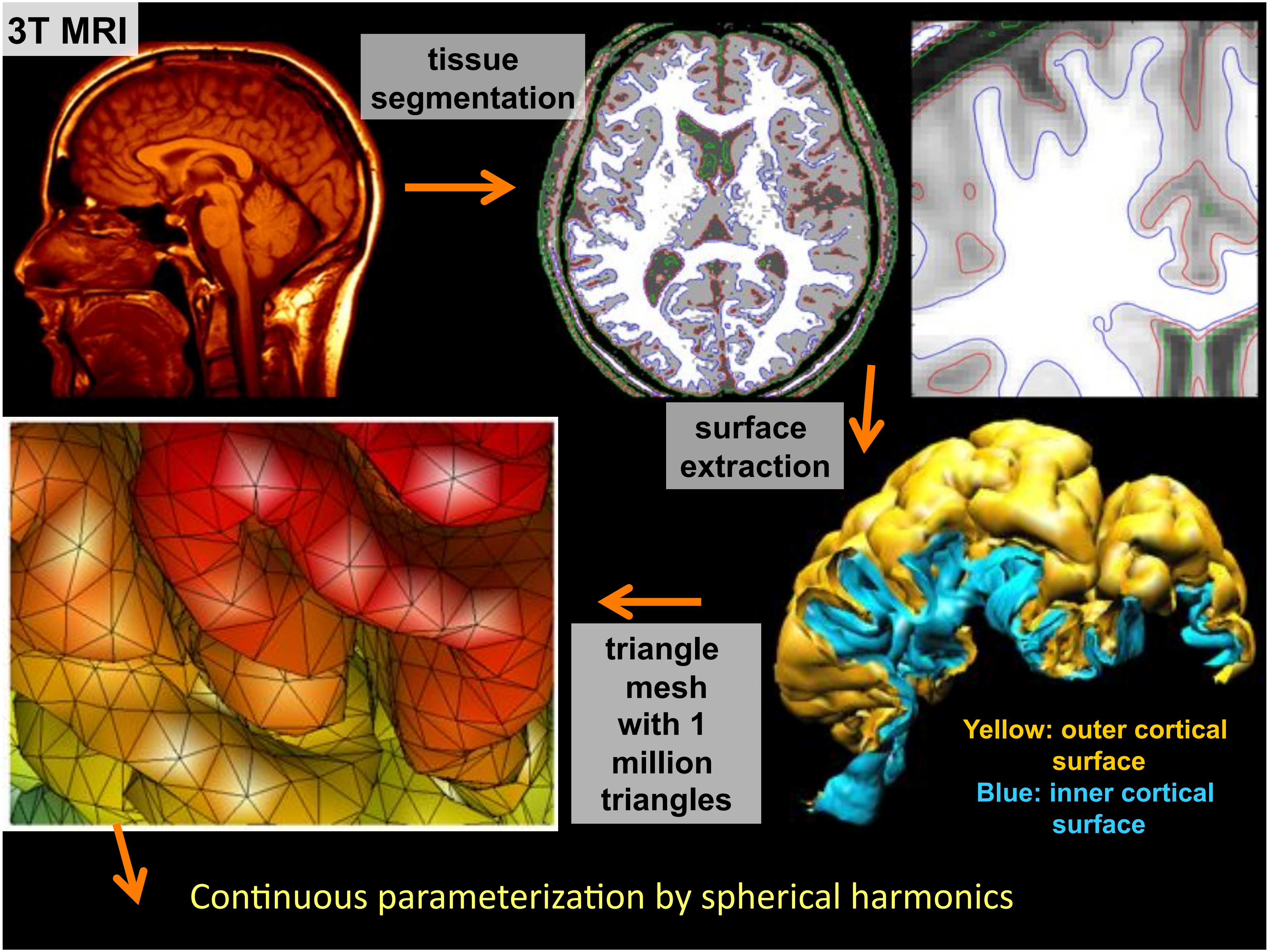
Final surface extraction result



Inner surface



Outer surface


3T MRI
tissue segmentation
surface extraction
triangle mesh with 1 million triangles
Yellow: outer cortical surface
Blue: inner cortical surface
Continuous parameterization by spherical harmonics

Data structure for polygonal mesh (autism_cortical_surface.mat)

Coordinates for subject 1

Vertex	1	2	3	4	5	6	….	40962
x	57.1876	41.0450	-53.1115	-38.1080	1.8440	-0.2458		
y	21.6388	-56.3448	29.8912	-65.5394	22.9715	9.4176		
z	2.9667	21.1399	-5.5088	23.6724	21.5146	16.9014		
Thickness	5.0	4.9	3.0	2.1	3.4	4.5		

Coordinates for subject 2

Vertex	1	2	3	4	5	6	….	40962
x	53.4240	41.0552	-61.4073	-43.2099	1.6256	-3.9101		
y	22.5535	-56.7731	20.9221	-65.9948	22.7979	29.7043		
z	7.1866	22.4754	-0.1368	21.3962	20.2838	-10.8959		
Thickness	5.5	3.4	2.7	5.1	3.7	4.5		

Corresponding vertices have approximate anatomical homology.

Quadratic surface fitting

$$s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots$$

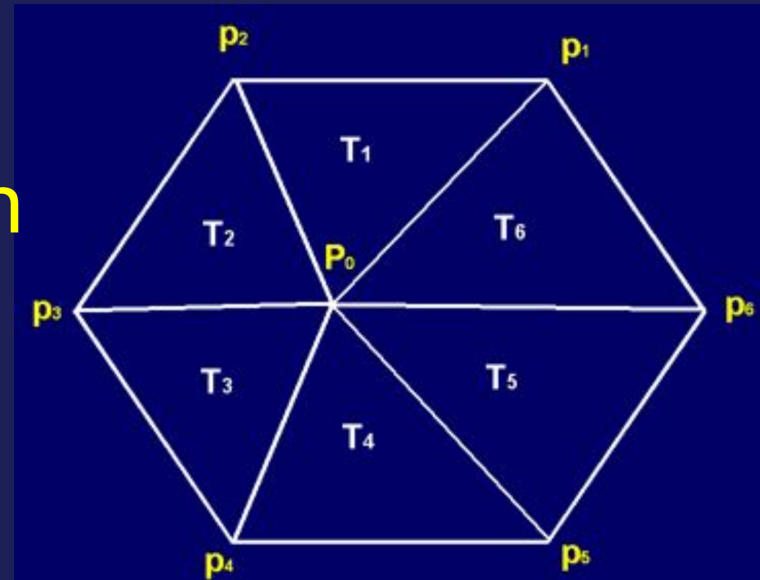
Riemannian metric tensors

$$g = \begin{pmatrix} 1+\beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & 1+\beta_2^2 \end{pmatrix} \qquad l = \begin{pmatrix} \beta_3 & \beta_4 \\ \beta_4 & \beta_5 \end{pmatrix}$$

Mean curvature

$$K_M = \frac{\mathrm{tr}(g^{-1}l)}{2} = \frac{\beta_3(1+\beta_2^2) + \beta_5(1+\beta_1^2) - 2\beta_1\beta_2\beta_4}{2 + 4(\beta_1^2 + \beta_2^2)}$$

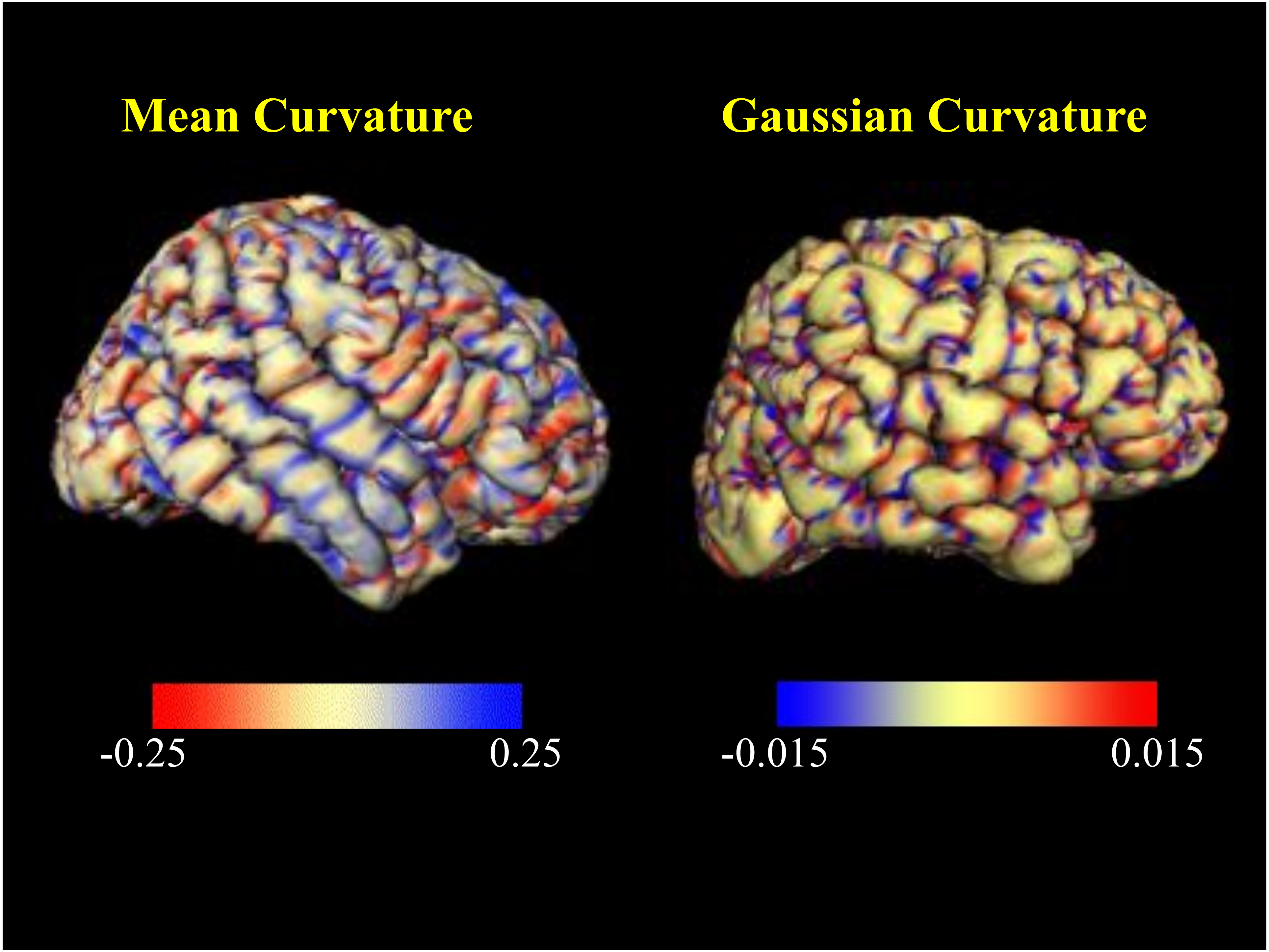
Polynomial Regression on irregular triangular mesh



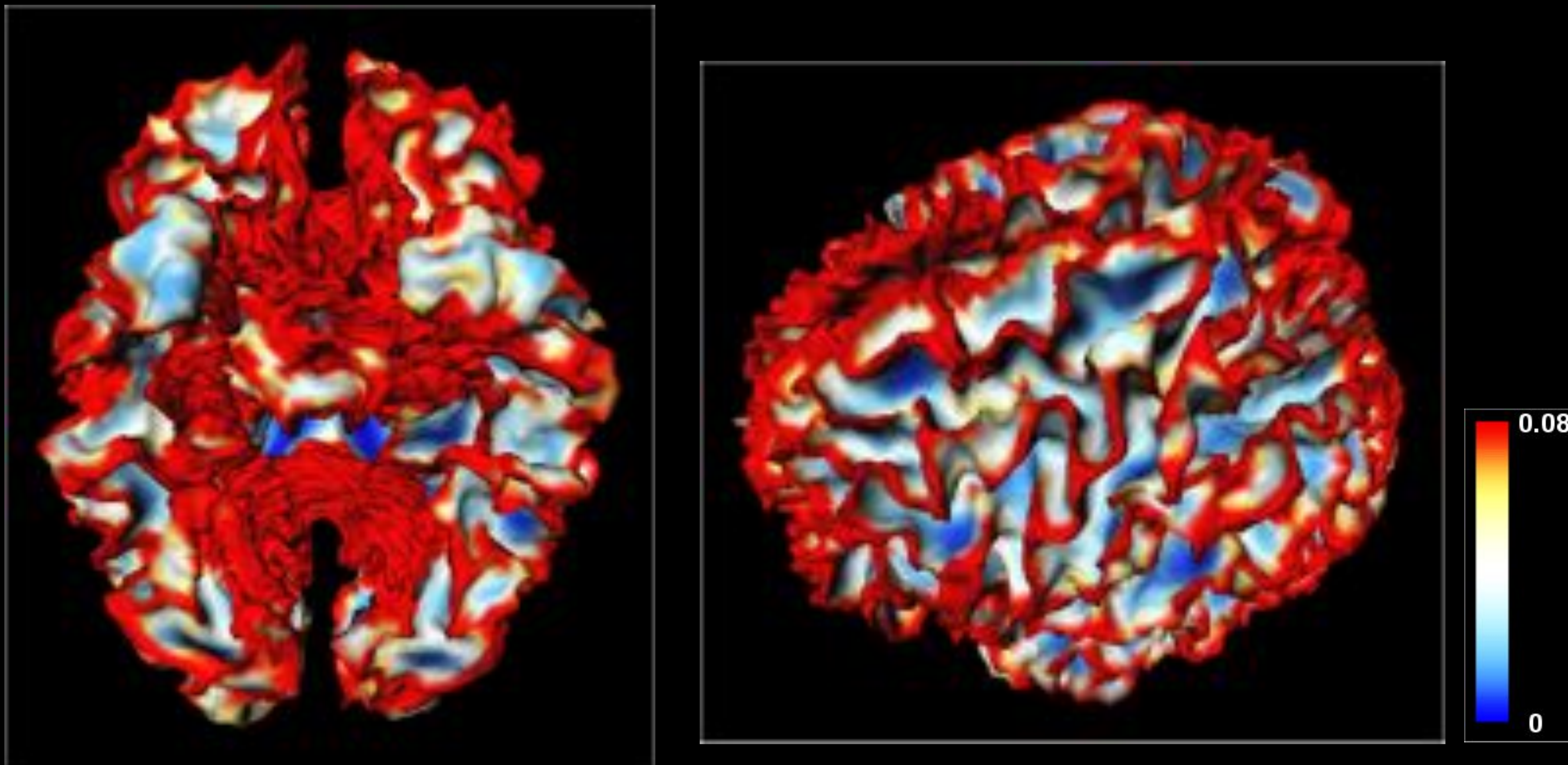
$$Y = \mathbb{X}\beta$$

$$\begin{pmatrix} u_1^3 \\ u_2^3 \\ \vdots \\ u_m^3 \end{pmatrix} = \begin{pmatrix} u_1^1 & u_1^2 & (u_1^1)^2 & u_1^1 u_1^2 & (u_1^2)^2 \\ u_2^1 & u_2^2 & (u_2^1)^2 & u_2^1 u_2^2 & (u_2^2)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_m^1 & u_m^2 & (u_m^1)^2 & u_m^1 u_m^2 & (u_m^2)^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

$$\hat{\beta} = (\mathbb{X}'\mathbb{X})^{-}\mathbb{X}'Y$$

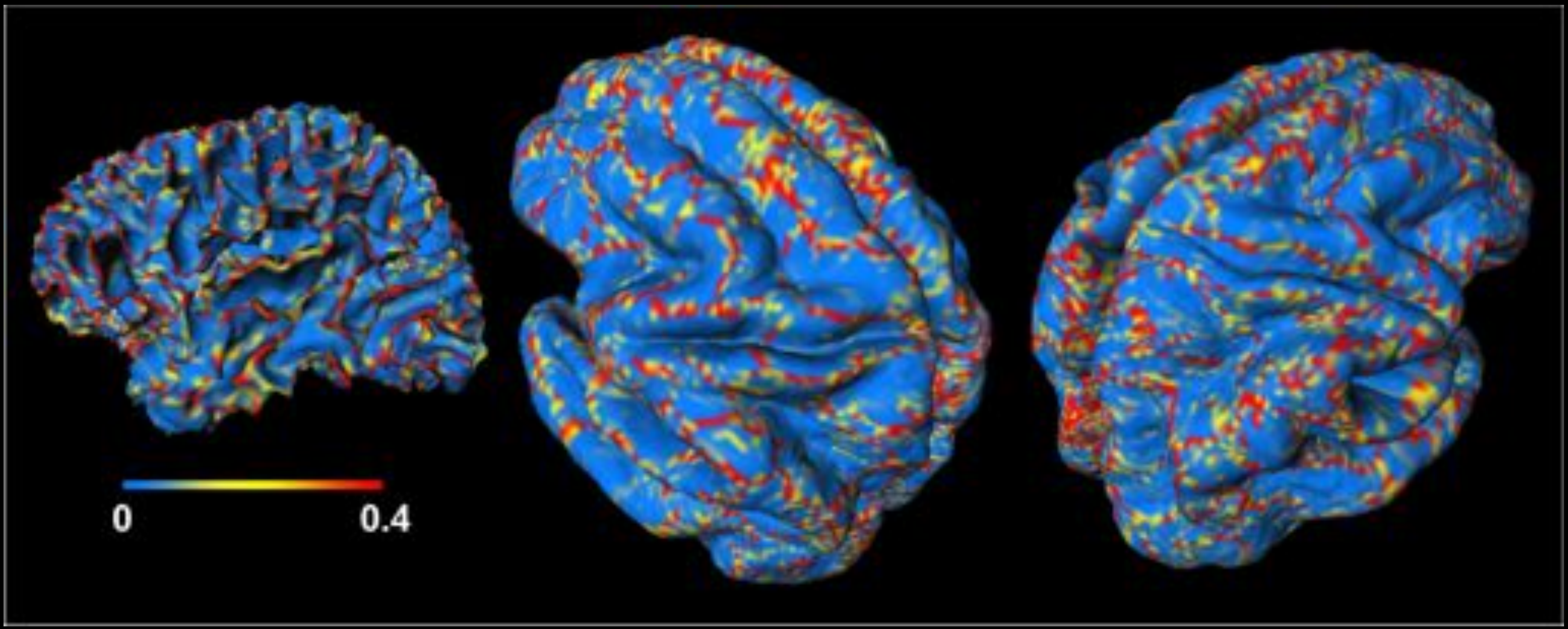

Mean Curvature
Gaussian Curvature
-0.25
0.25
-0.015
0.015

Bending Energy for 14 year old subject



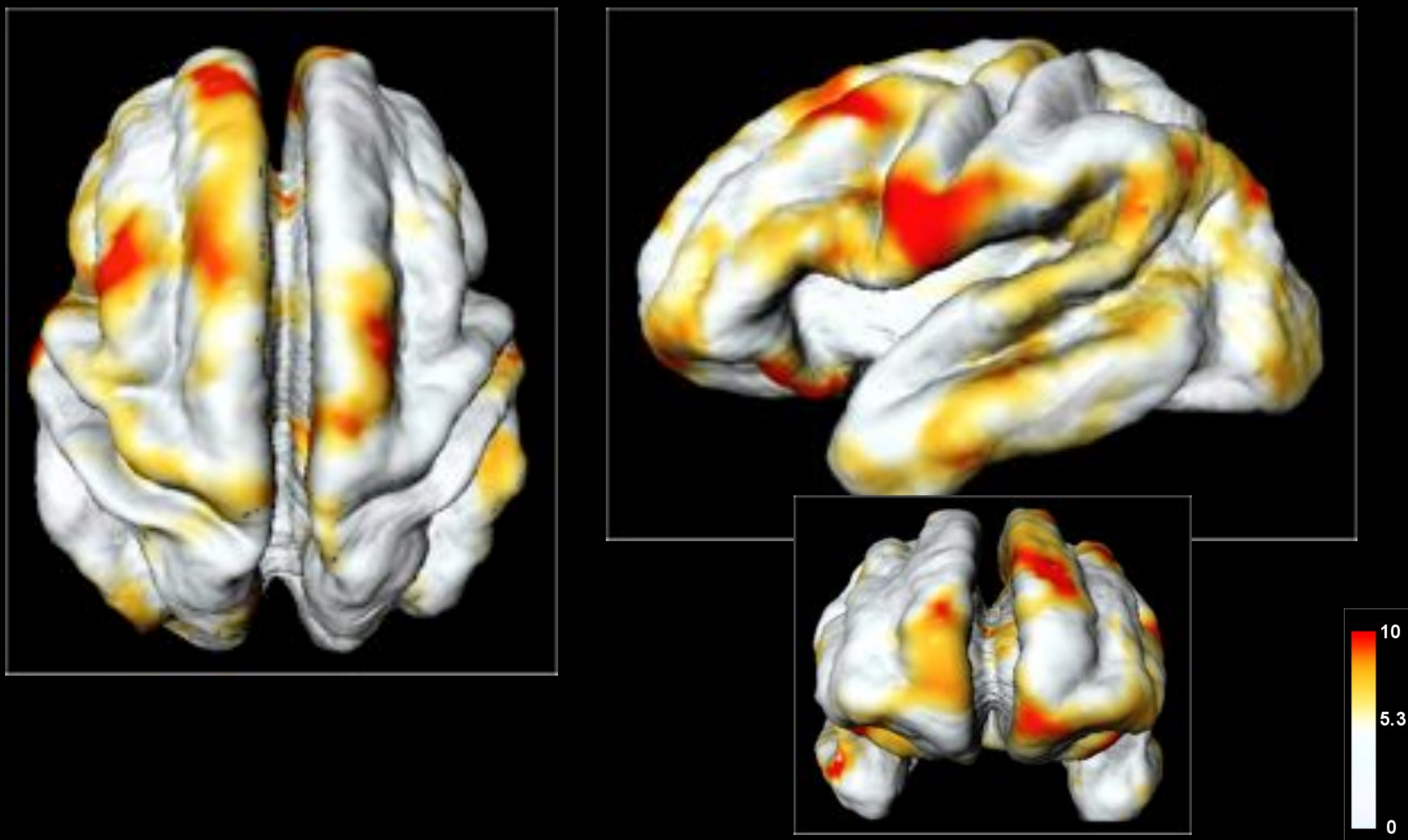
Bending energy or thin-plate spline energy can be used to measure the curvature of the surface.
Between ages 12 and 16, it increases both locally and globally.

CVPR 2003



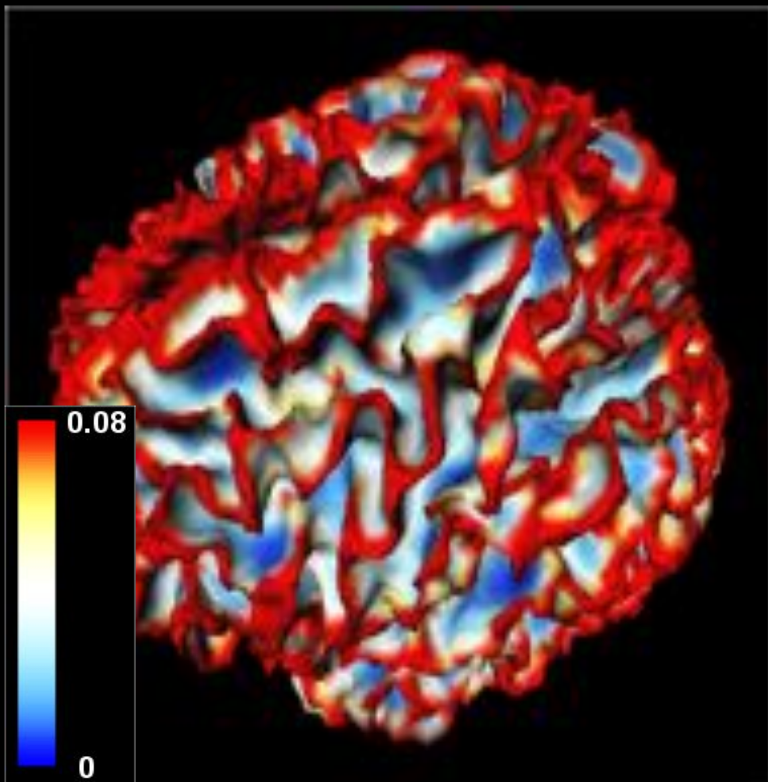
Principle curvature maps projected on the average template

Curvature change *t* map between age 12 and 16

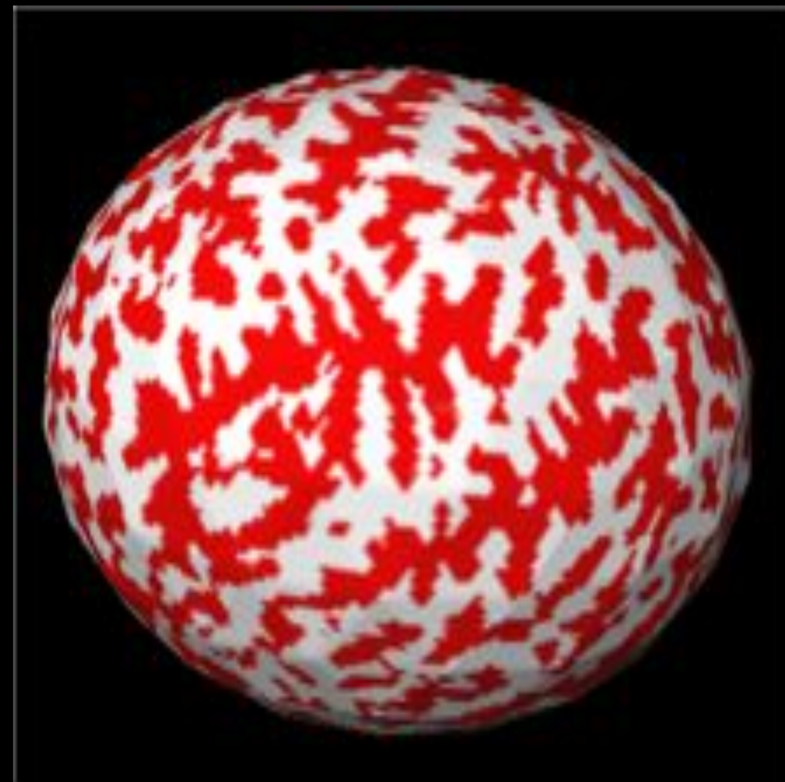


Compute cortical curvature and map curvature to unit sphere

3D problem

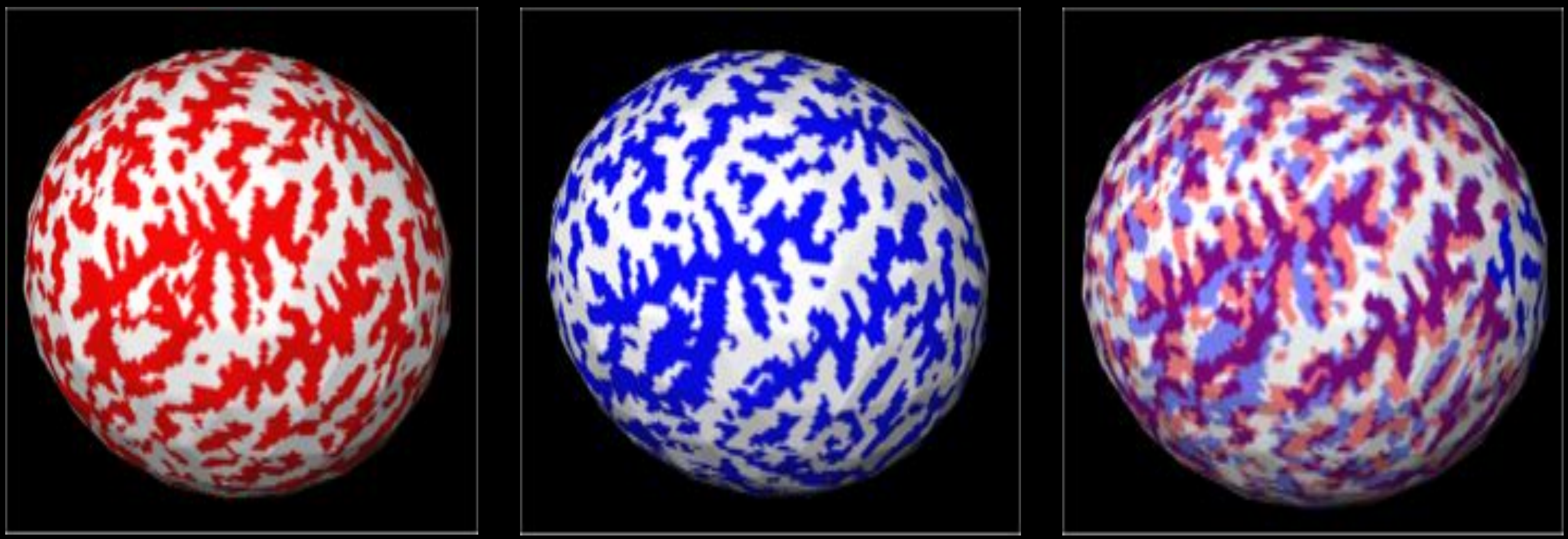


2D problem



Unit sphere gives a natural coordinate system (spherical coordinates).

Sulcal pattern matching



Misalignment

Sulcal pattern matching by minimizing

objective function = curvature difference - smoothness of deformation

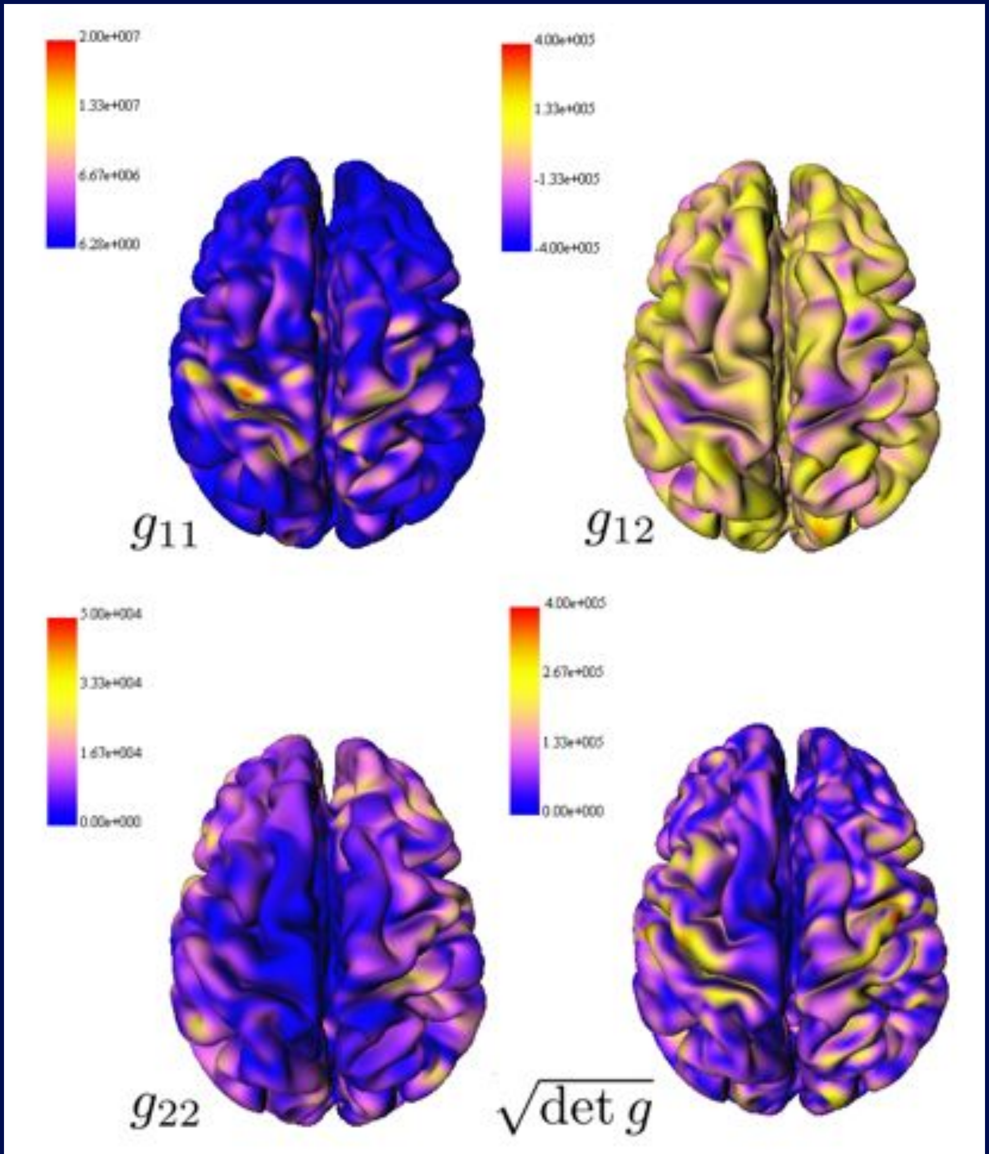
See Paul Thompson's earlier IEEE TMI paper

Surface area expansion/shrinking

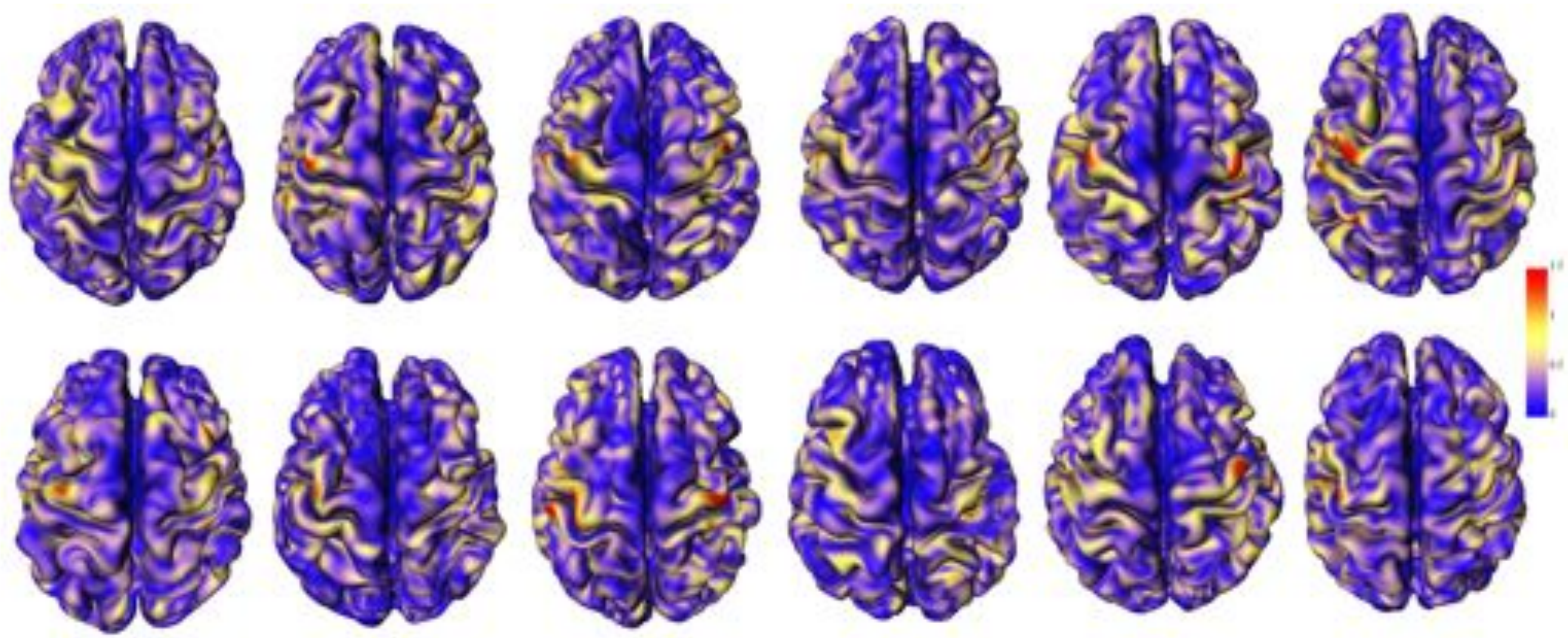
Local surface area element:

$$\sqrt{|g|} = \sqrt{1 + \beta_1^2 + \beta_2^2}$$

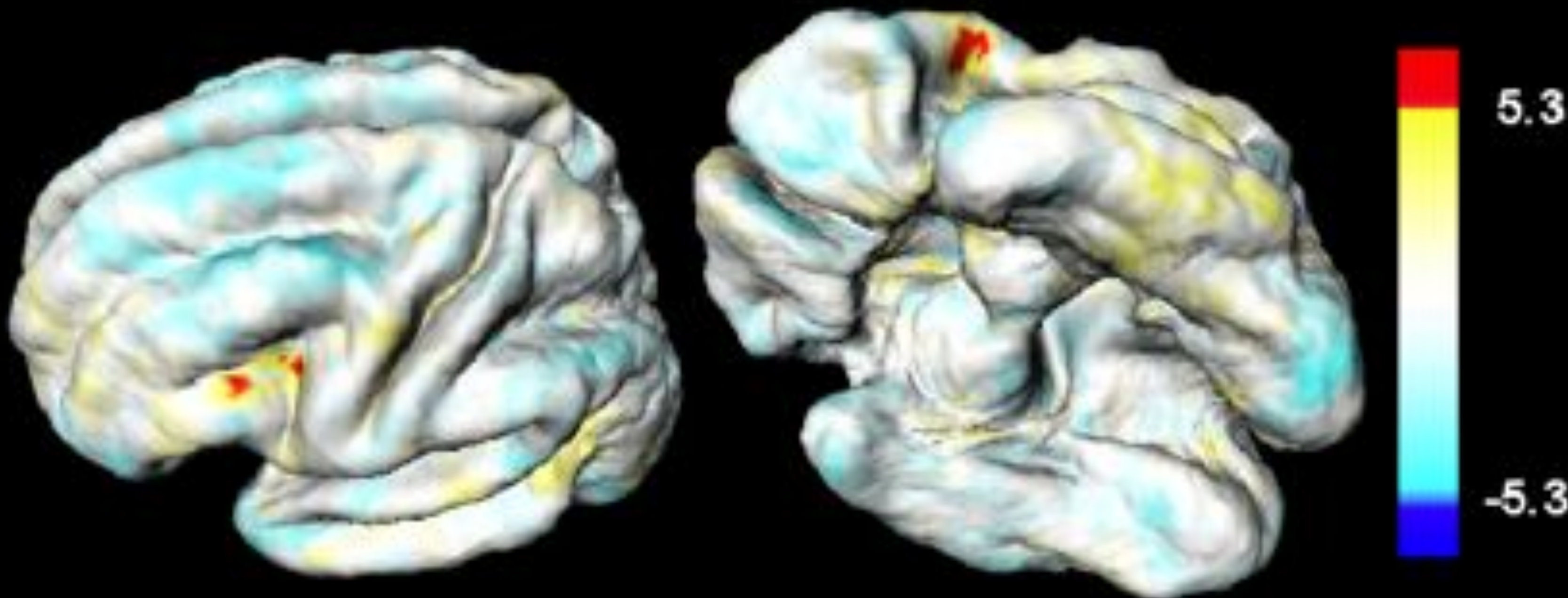
Spherical harmonic representation was used to analytically compute and smooth surface area element.



Local area expansion with respect to a template (it ranges between 0 and 1.3)



Surface area change *t* map



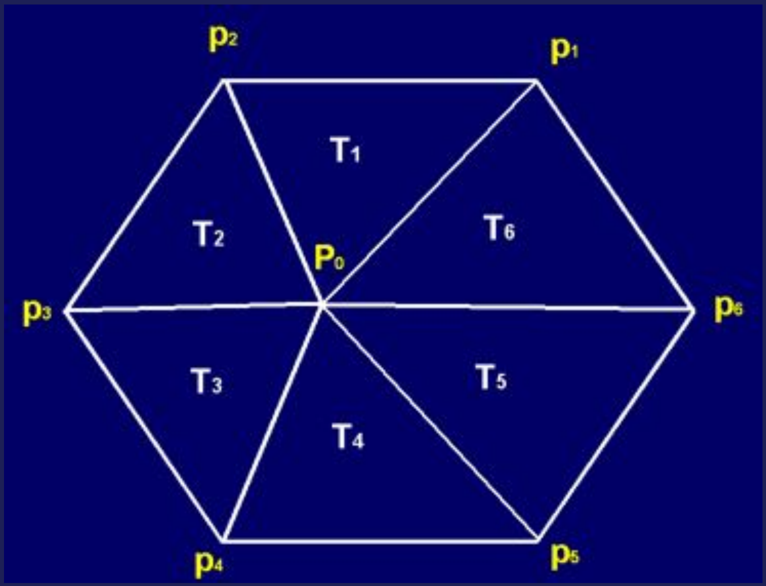
dilatation rate between age 12 and 16

min = - 57 %   mean = - 0.02 %   max = 65 %

Laplace-Beltrami Operator

$$\Delta_X F = \frac{1}{|g|^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u^i}\left(|g|^{1/2} g^{ij} \frac{\partial F}{\partial u^j}\right)$$

Estimating differential operator on manifolds



$$\widehat{\Delta} F(p_0) = w_0 F(p_0) + w_1 F(p_1) + \cdots + w_m F(p_m)$$

Estimation via conformal transformation

$$s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots$$

$$g = \begin{pmatrix} 1+\beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & 1+\beta_2^2 \end{pmatrix}$$

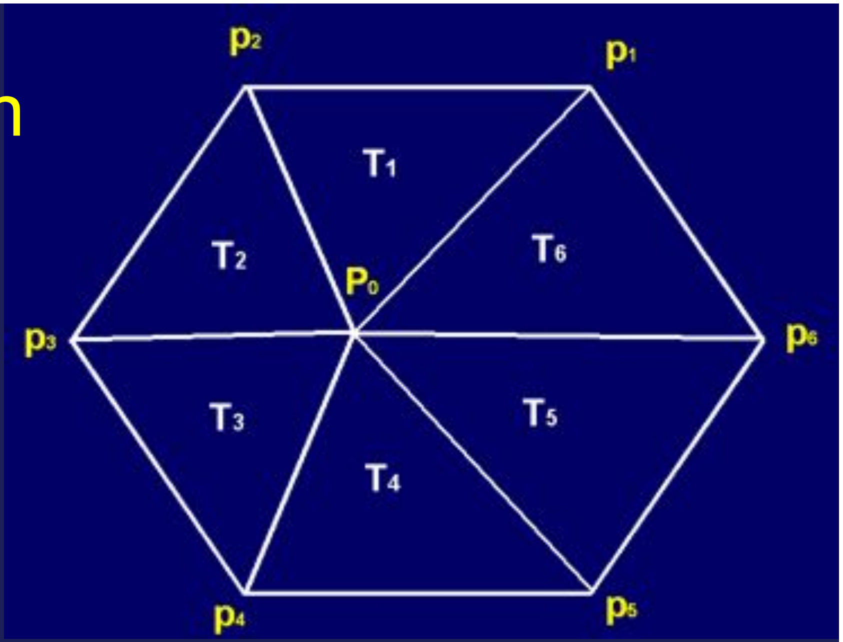
Laplace-Beltrami operator is invariant

$$s(v^1, v^2) = \gamma_1 (v^1)^2 + \gamma_2 v^1 v^2 + \gamma_3 (v^2)^2 + \cdots$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\Delta x = \frac{1}{\lambda}\left(\frac{\partial^2}{\partial^2 u^1} + \frac{\partial^2}{\partial^2 u^2}\right)}$$

Estimating the planar Laplacian on irregular triangular mesh



$$Y = \mathbb{X}\beta$$

$$\begin{pmatrix} F(p_1) - F(p_0) \\ F(p_2) - F(p_0) \\ \vdots \\ F(p_m) - F(p_0) \end{pmatrix} = \begin{pmatrix} v_1^1 & v_1^2 & (v_1^1)^2 & v_1^1 v_1^2 & (v_1^2)^2 \\ v_2^1 & v_2^2 & (v_2^1)^2 & v_2^1 v_2^2 & (v_2^2)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_m^1 & v_m^2 & (v_m^1)^2 & v_m^1 v_m^2 & (v_m^2)^2 \end{pmatrix} \begin{pmatrix} \frac{\partial F(p_0)}{\partial v^1} \\ \frac{\partial F(p_0)}{\partial v^2} \\ \frac{\partial^2 F(p_0)}{\partial (v^1)^2} \\ \frac{\partial^2 F(p_0)}{\partial v^1 \partial v^2} \\ \frac{\partial^2 F(p_0)}{\partial (v^2)^2} \end{pmatrix}$$

Thin Plate Spline Parameterization

Measurement f is represented as

$$f(p) = \sum_i \alpha_i \phi_i(p) + \sum_j \beta_j \varphi(p - p_j)$$

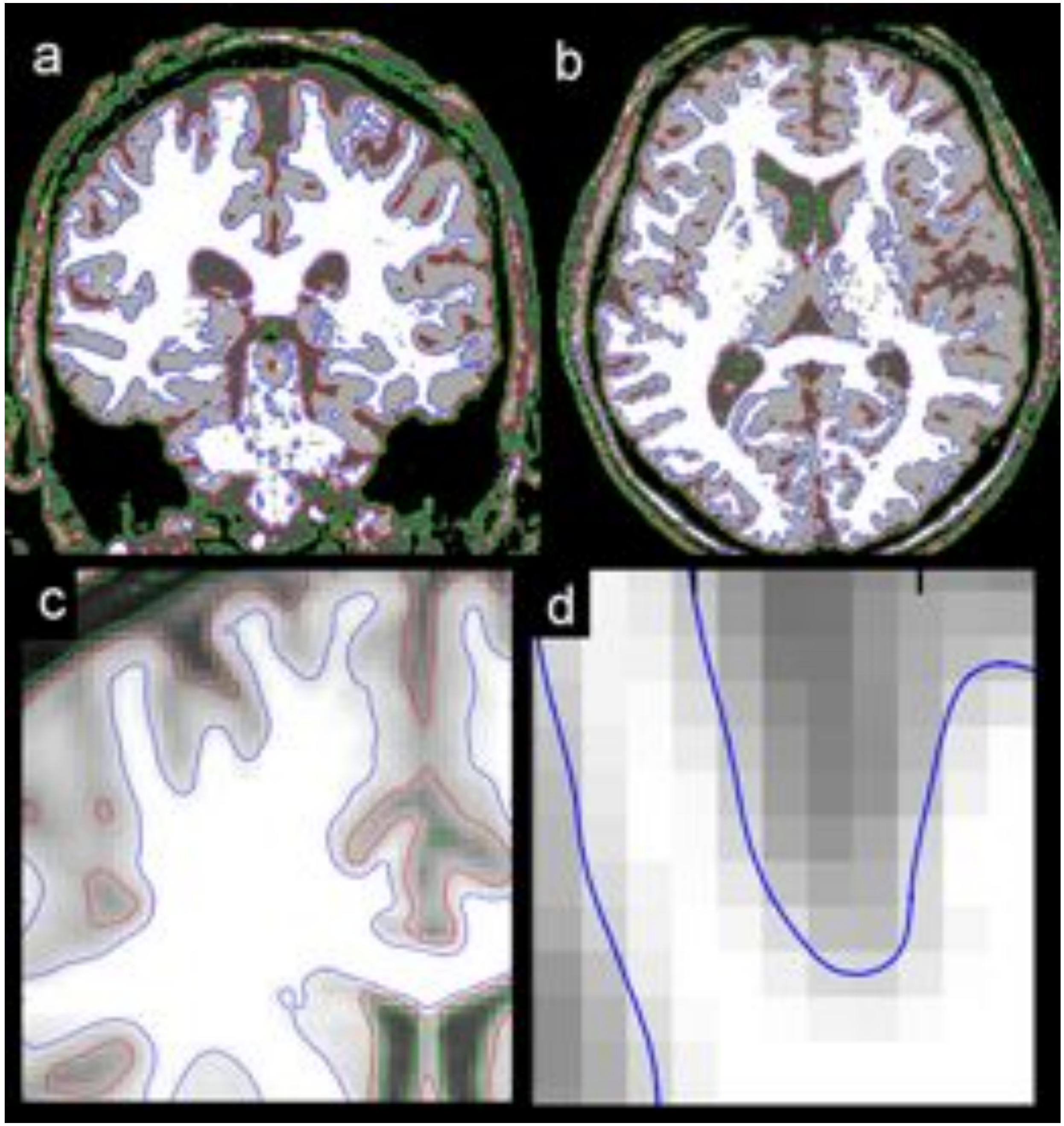
where ϕ_i is polynomial basis and φ is the TPS radial basis

Parameters are estimated by minimizing

$$\min_f \sum_{i=1} \left| y_i - f(p_i) \right|^2 + \lambda J_3^2(f),$$

Thin Plate Spline (TPS) segmentation and modeling

TPS represents anatomical boundary as the zero level set of smooth function consists of polynomial and radial basis functions (Wahba, 1990; Xie et al., 2005a).



Spherical Harmonic (SPHARM) Representation

- Spherical harmonics are basis functions on a unit sphere.
- SPHARM can be used to construct the Fourier series expansion of a functional measurement
- SPAHRM has been used in parameterizing anatomical boundary
- New more localized approaches: wavelets, weighted-SPAHRM

Spherical harmonics

Y_{lm} is called the *spherical harmonic* of degree l and order m .

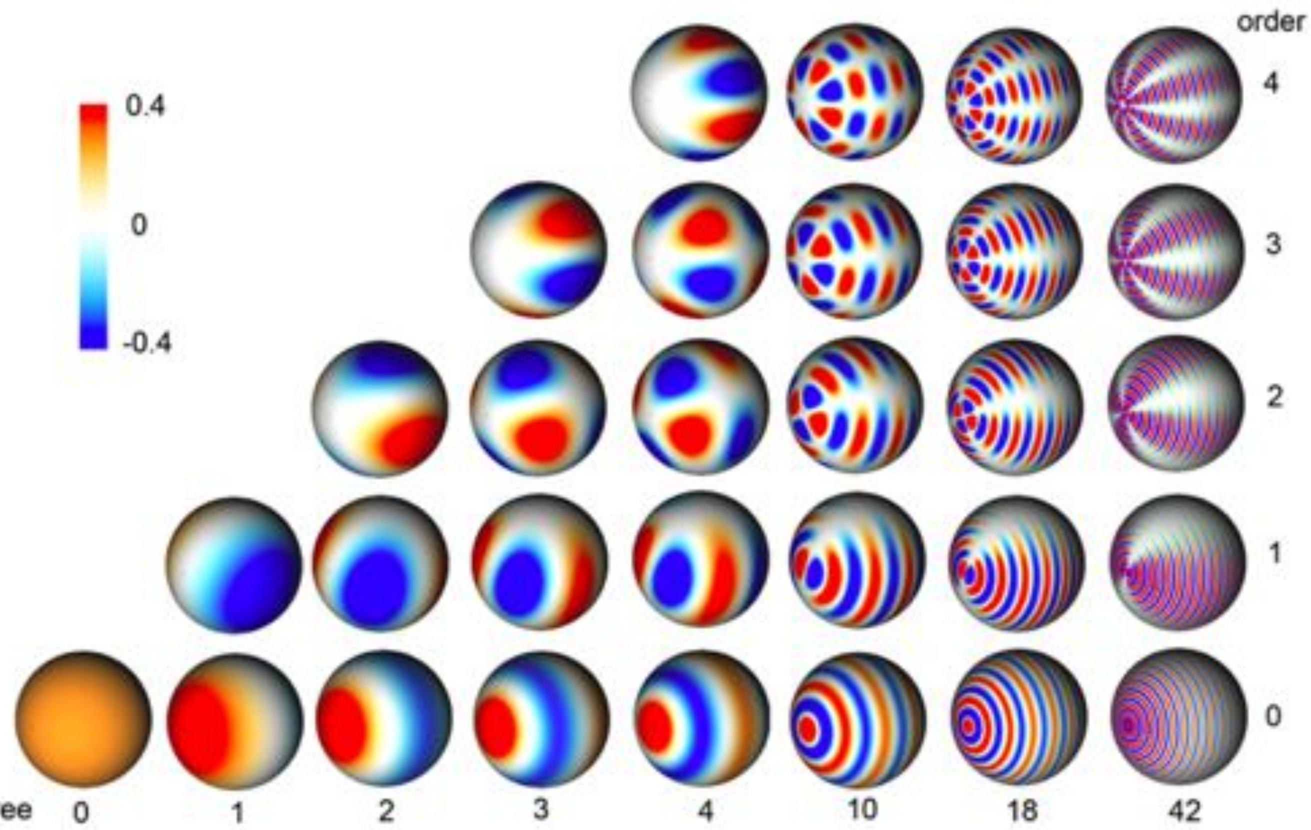
$$Y_{lm} = \begin{cases} c_{lm} P_l^{|m|}(\cos\theta)\sin(|m|\varphi), & -l \le m \le -1, \\ \frac{c_{lm}}{\sqrt{2}} P_l^0(\cos\theta), & m = 0, \\ c_{lm} P_l^{|m|}(\cos\theta)\cos(|m|\varphi), & 1 \le m \le l, \end{cases}$$

where $c_{lm} = \sqrt{\frac{2l+1}{2\pi}\frac{(l-|m|)!}{(l+|m|)!}}$ and P_l^m is the associated Legendre polynomials of order m.

Spherical harmonic of degree *l* and order *m*

$$Y_{lm} = \begin{cases} c_{lm} P_l^{|m|}(\cos\theta)\sin(|m|\varphi), & -l \le m \le -1, \\ \frac{c_{lm}}{\sqrt{2}} P_l^0(\cos\theta), & m = 0, \\ c_{lm} P_l^{|m|}(\cos\theta)\cos(|m|\varphi), & 1 \le m \le l, \end{cases}$$

Coarse anatomical detail



Fine anatomical detail

Coarse detail ↔ **Fine detail**

SPHRM representation

•Given functional measurement *f(p)* on a unit sphere, it is modeled as

$$f(p) = \sum_{l=0}^{k} \sum_{m=-l}^{l} f_{lm} Y_{lm}(p) + e(p)$$

e: noise (image processing, numerical, biological)

f_{lm}: unknown Fourier coefficients

•The parameters are estimated in the least squares fashion.

- For measurements $f(p_1), f(p_2), \cdots, f(p_n)$, $(n > 46,000)$, we set up normal equations:

$$f(p_i) = \sum_{l=0}^{k} \sum_{m=-l}^{l} \beta_{lm} Y_{lm}(p_i).$$

i-th mesh vertex

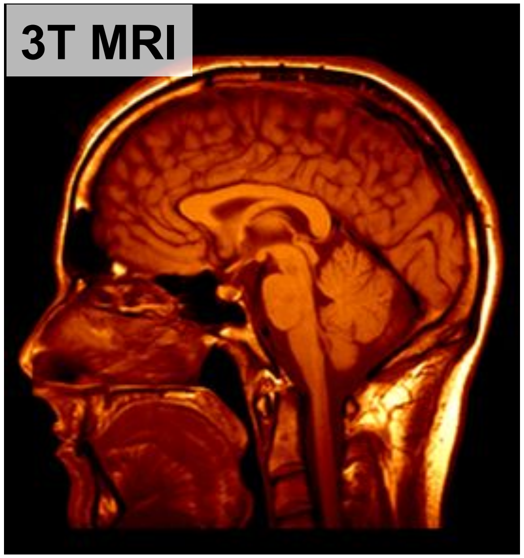
- Matrix form:

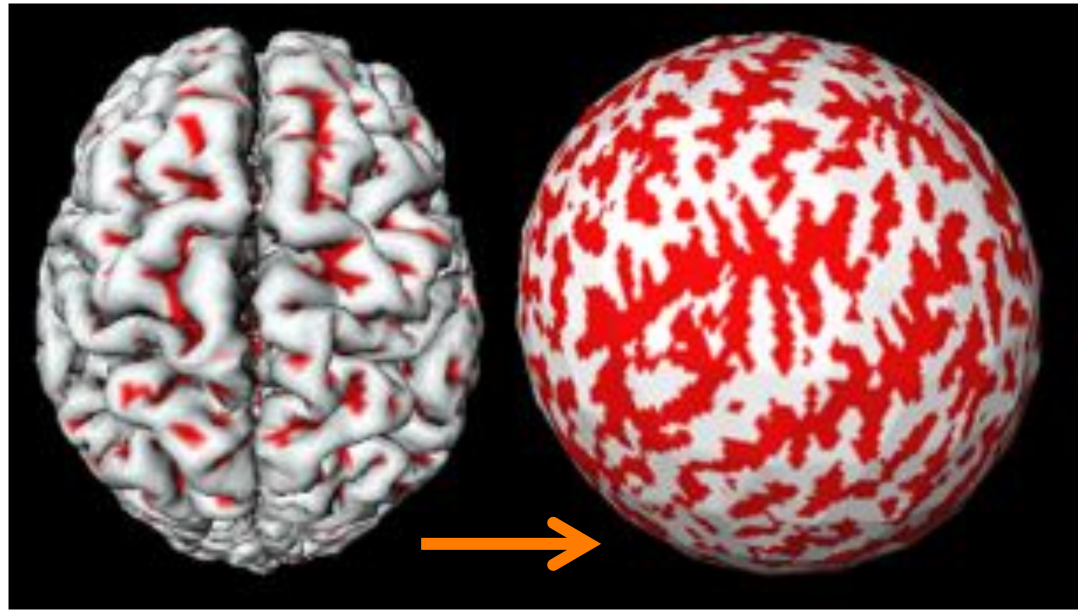
$$\underbrace{\begin{pmatrix} f(p_1) \\ f(p_2) \\ \vdots \\ f(p_n) \end{pmatrix}}_{\mathbf{F}} = \underbrace{\begin{pmatrix} Y_{00}(p_1) & Y_{1-1}(p_1) & Y_{10}(p_1) & \cdots & Y_{kk}(p_1) \\ Y_{00}(p_2) & Y_{1-1}(p_2) & Y_{10}(p_2) & \cdots & Y_{kk}(p_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{00}(p_n) & Y_{1-1}(p_n) & Y_{10}(p_n) & \cdots & Y_{kk}(p_n) \end{pmatrix}}_{\mathbf{Y}} \underbrace{\begin{pmatrix} \beta_{00} \\ \beta_{1-1} \\ \vdots \\ \beta_{kk} \end{pmatrix}}_{\beta}$$

40962 x 7000

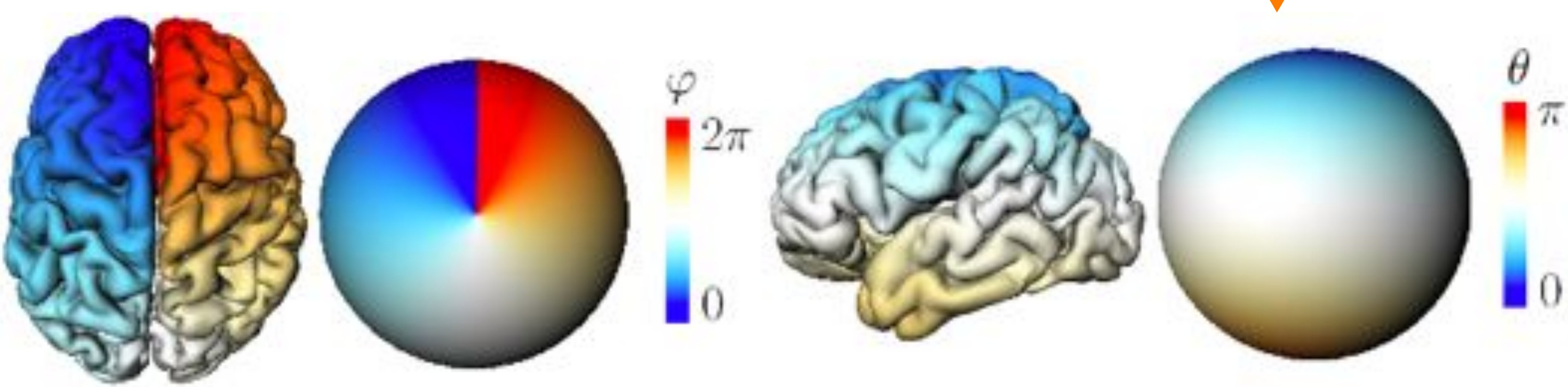
Estimation: $\widehat{\beta} = (\mathbf{Y}'\mathbf{Y})^{-1}\mathbf{Y}'\mathbf{F}.$

Cortical Surface Modeling





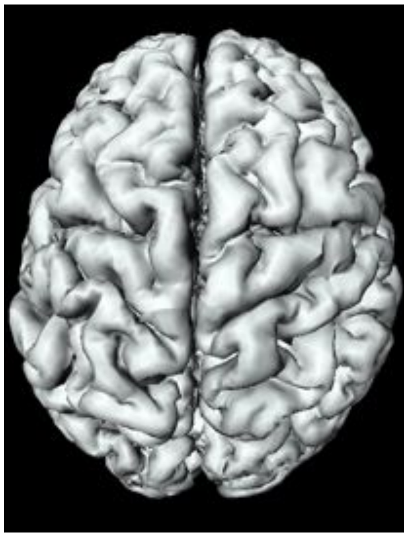
Deformable surface algorithm

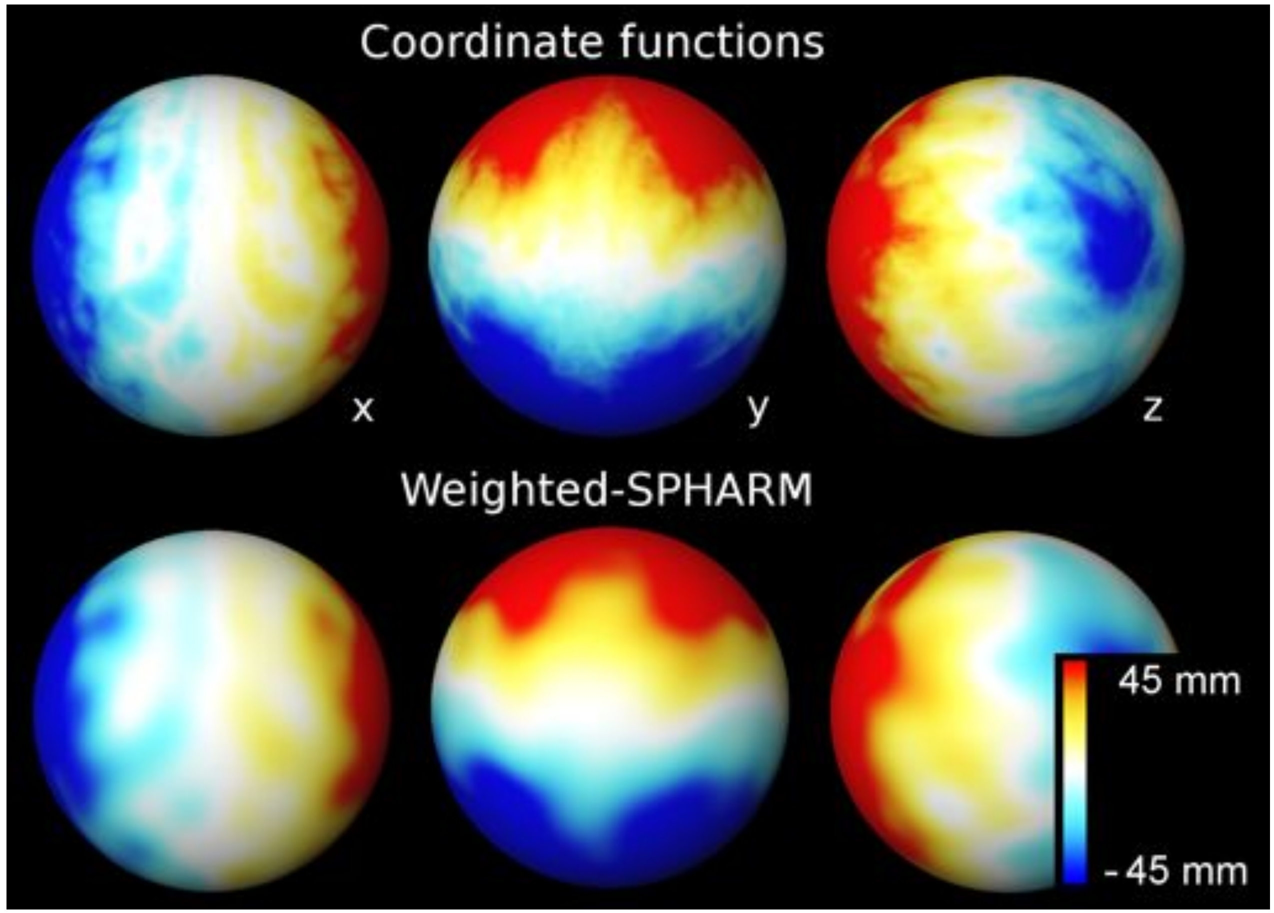


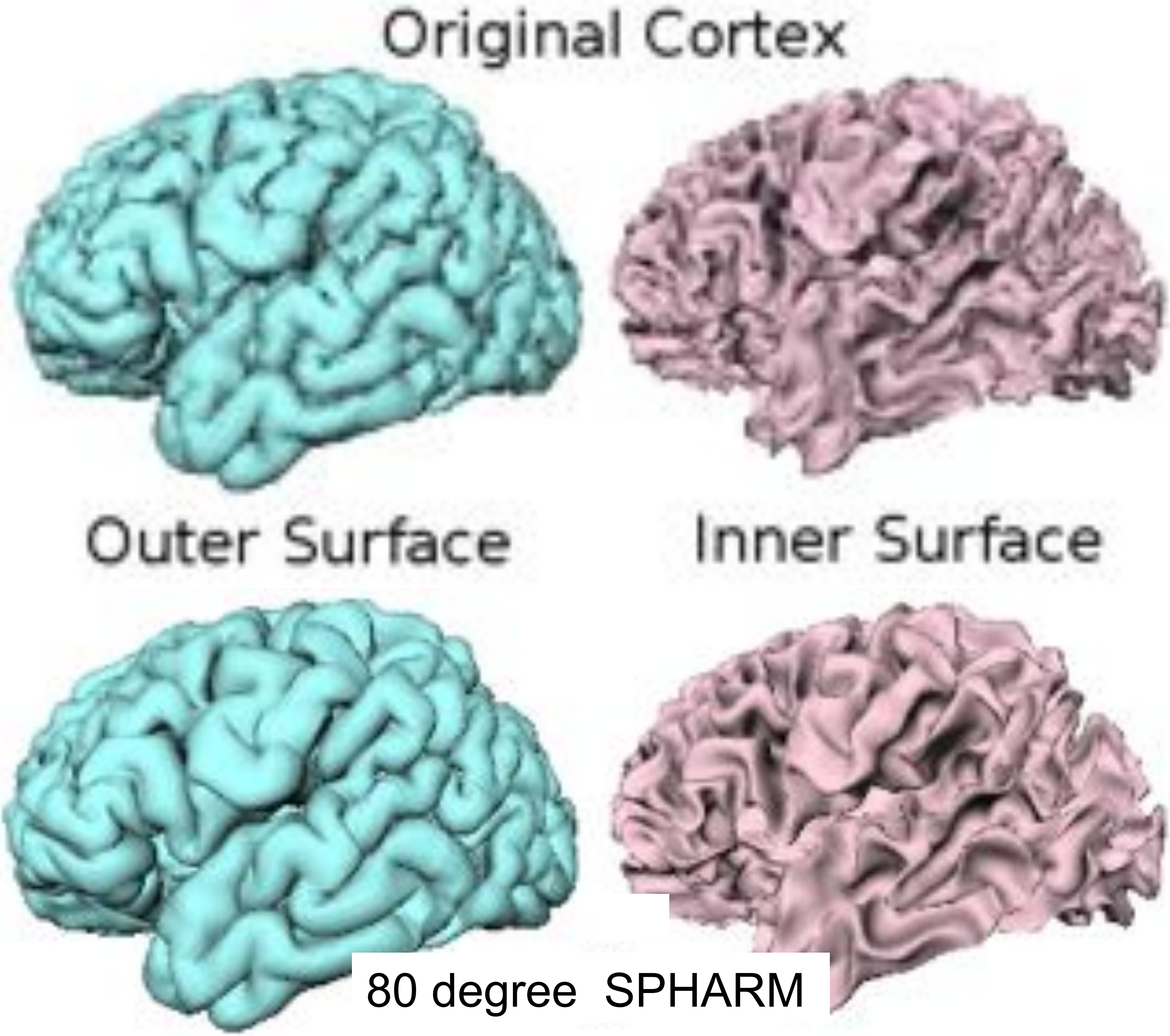
Spherical angle based coordinate system

Mapping from cortex to unit sphere

Each x, y, z Cartesian coordinates are represented independently.

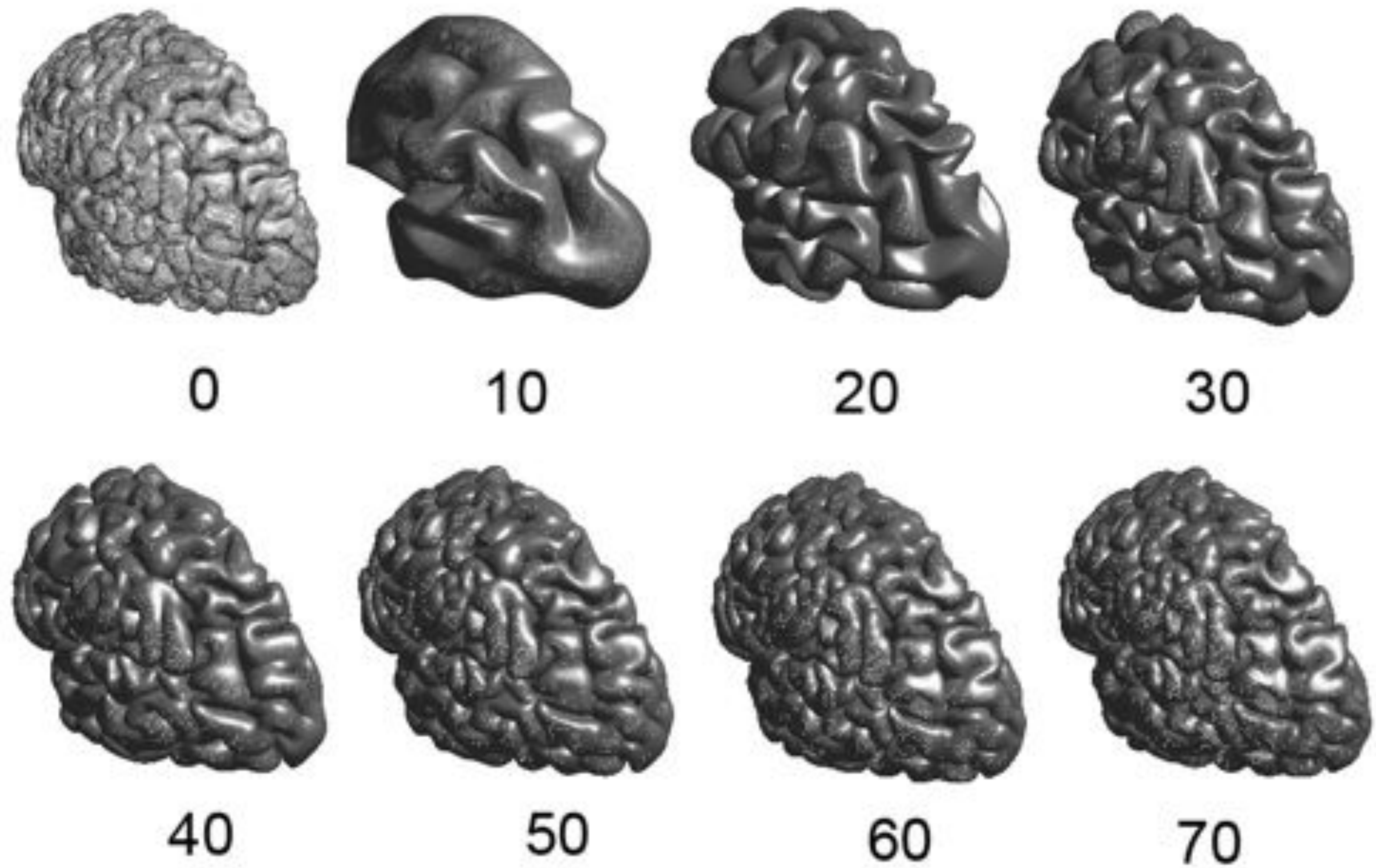




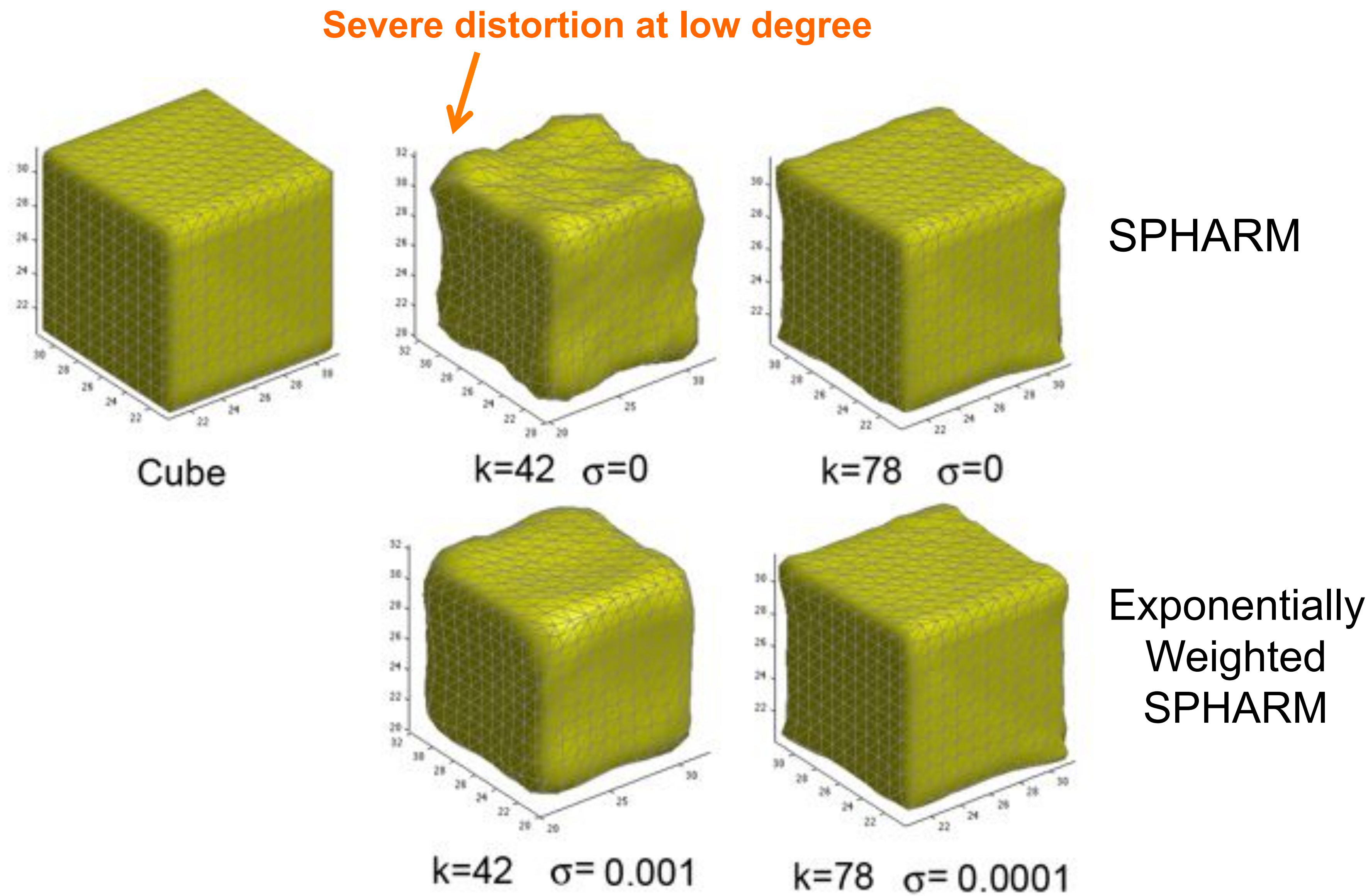


80 degree SPHARM

FreeSurfer results



Gibbs phenomenon (ringing artifacts) on surface



Determining the optimal degree via stepwise forward model selection framework

Consider the following $(k-1)$-th degree model

$$f(p_i) = \sum_{l=0}^{k-1} \sum_{m=-l}^{l} e^{-\lambda(l+1)\sigma} f_{lm} Y_{lm}(p_i) + \epsilon(p_i), \; i = 1, \cdots, n$$

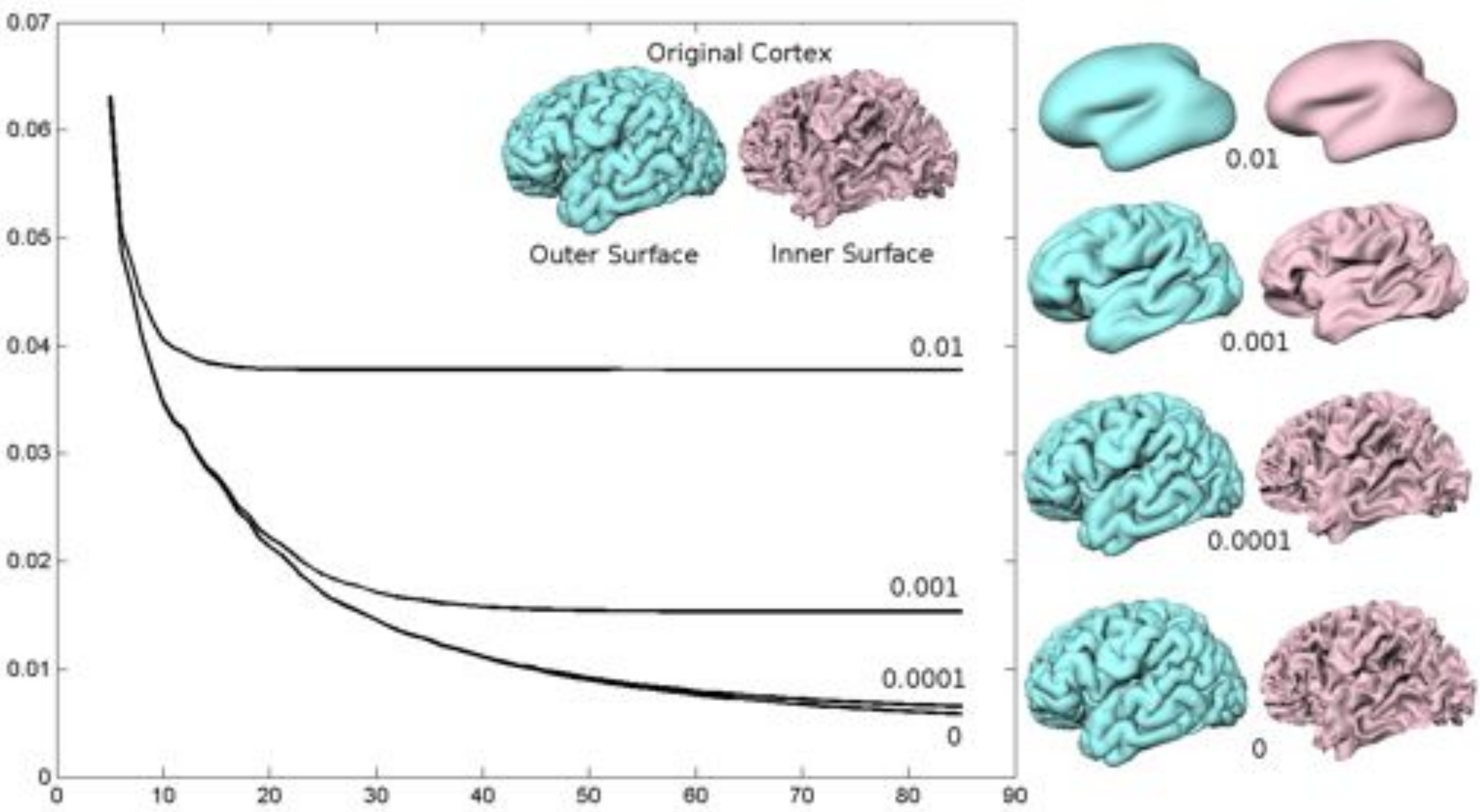
where ϵ are Gaussian random variables. Testing if the k-th degree model is better than the previous $(k-1)$-th degree model can be done by testing

$$H_0 : f_{km} = 0 \text{ for all } -k \le m \le k.$$

Then under the null hypothesis, the test statistic is

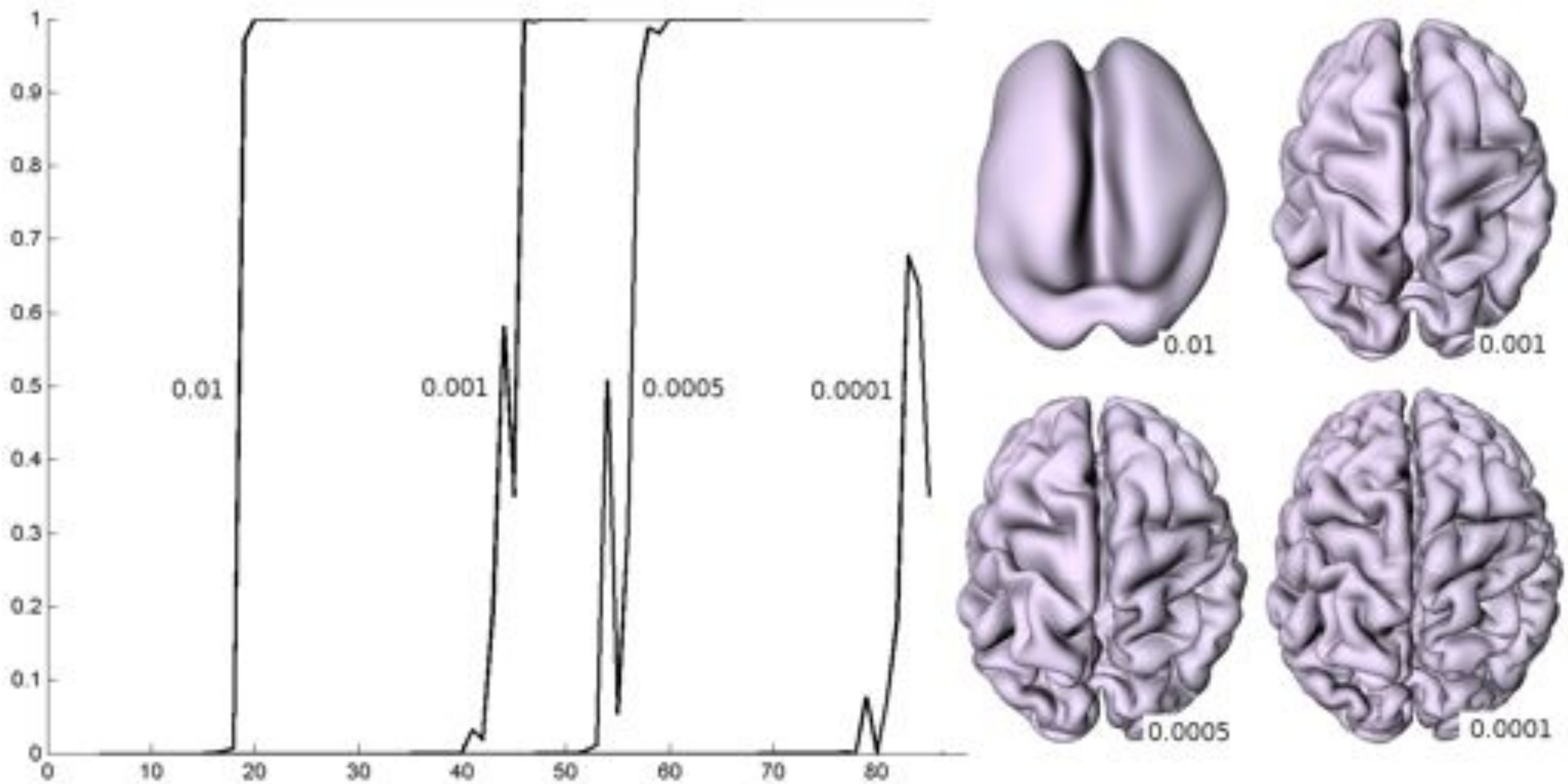
$$F = \frac{(\text{SSE}_{k-1} - \text{SSE}_k)/(2k+1)}{\text{SSE}_{k-1}/(n-(k+1)^2)} \sim F_{2k+1, n-(k+1)^2}$$

Weighted-SPHARM at the 80th degree for different bandwidth



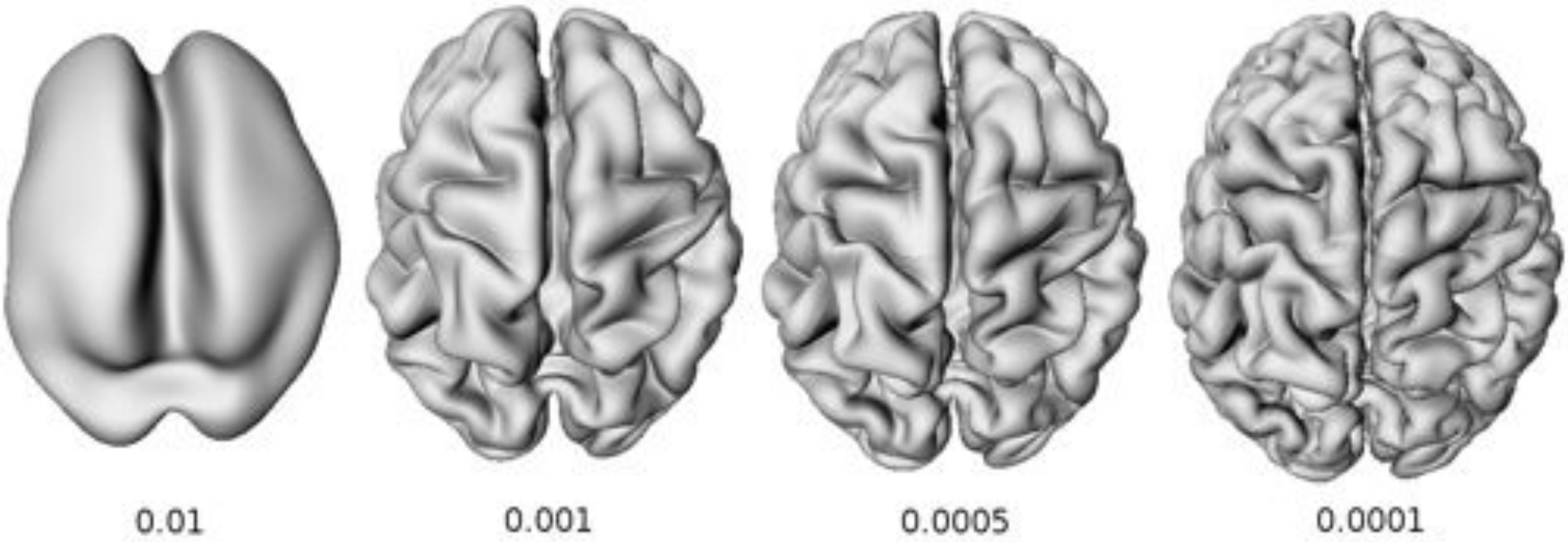
Root mean squared error (RMSE)
= error between original surface and weighted-SPHARM

For each bandwidth σ , optimal degree is automatically selected via **forward best model selection procedure**.



Optimal degree= first P-value >0.05

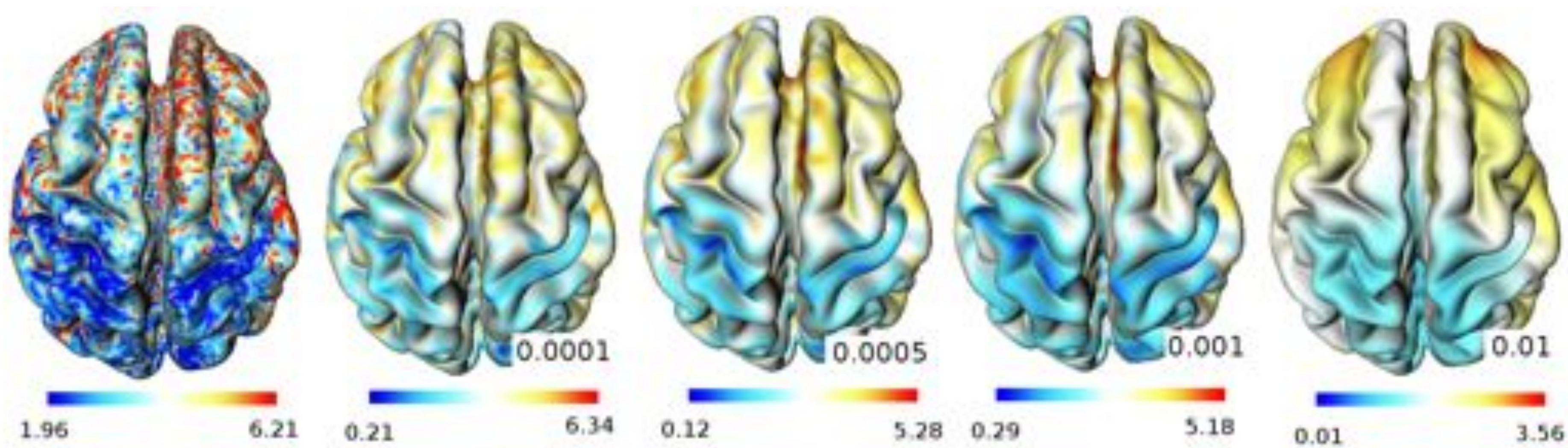
Weighted-SPHARM at different bandwidth



- The degree is selected automatically.

- The only free parameter in the model is the bandwidth.

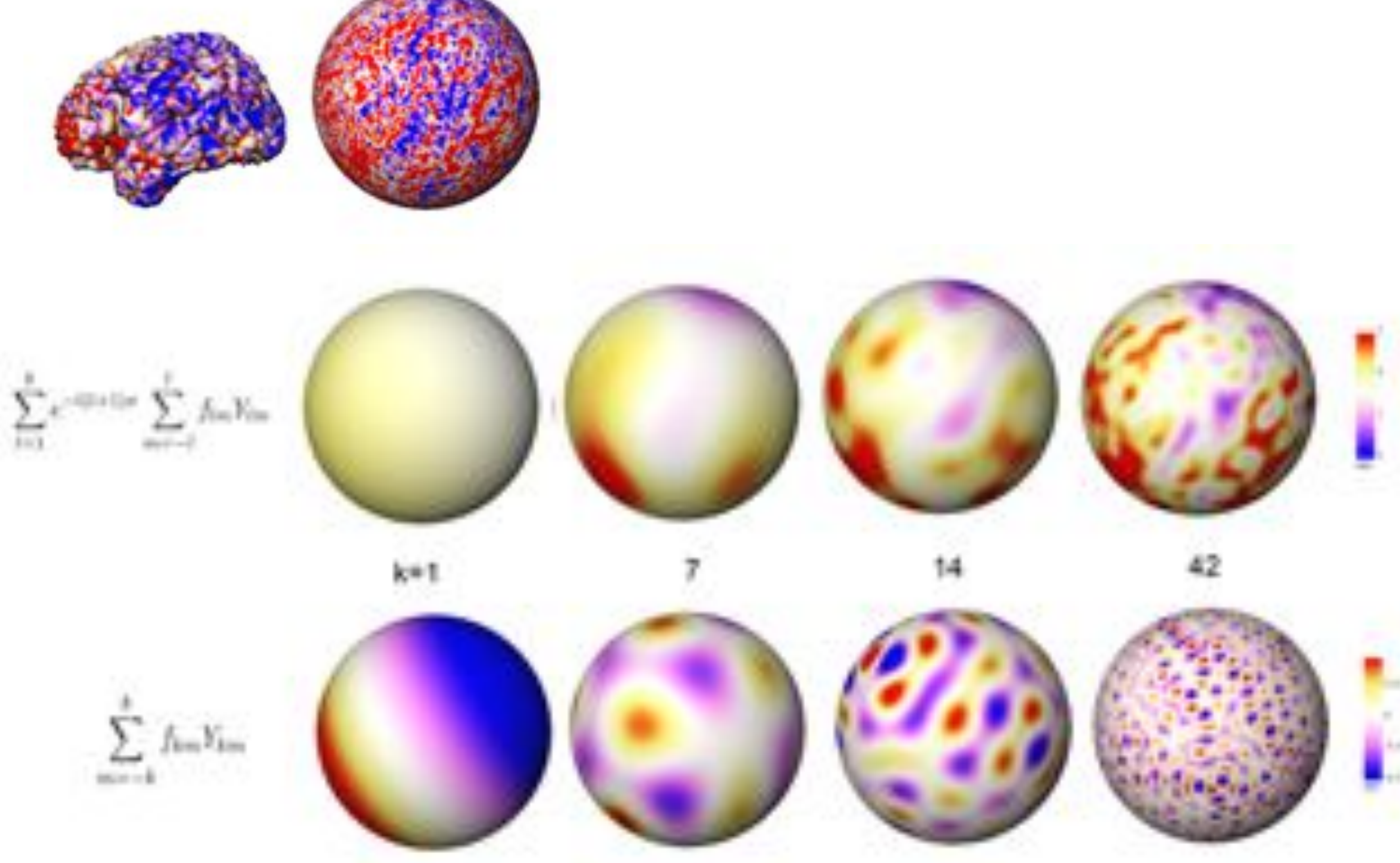
SPHARM estimation of cortical thickness



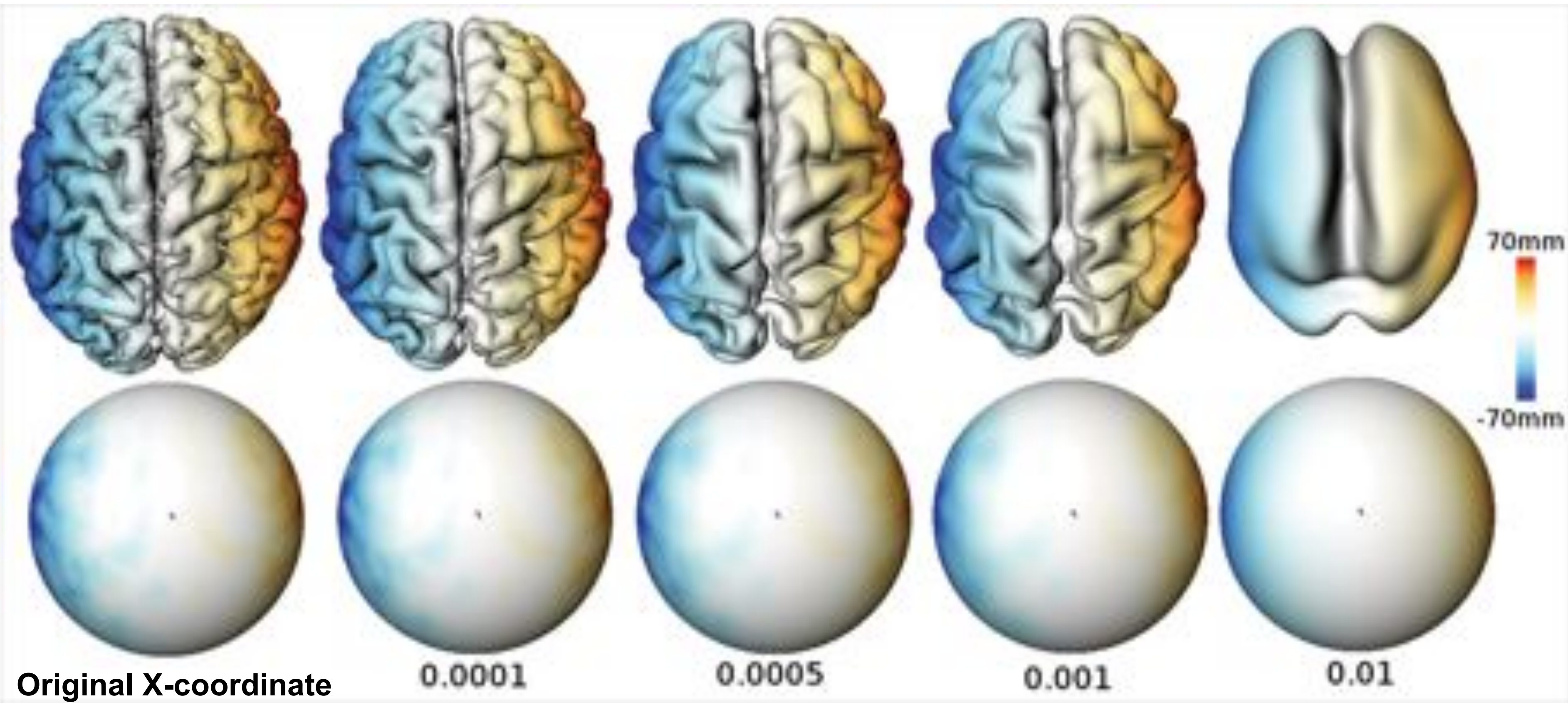
Thickness estimation based on traditional method

Too much smoothing

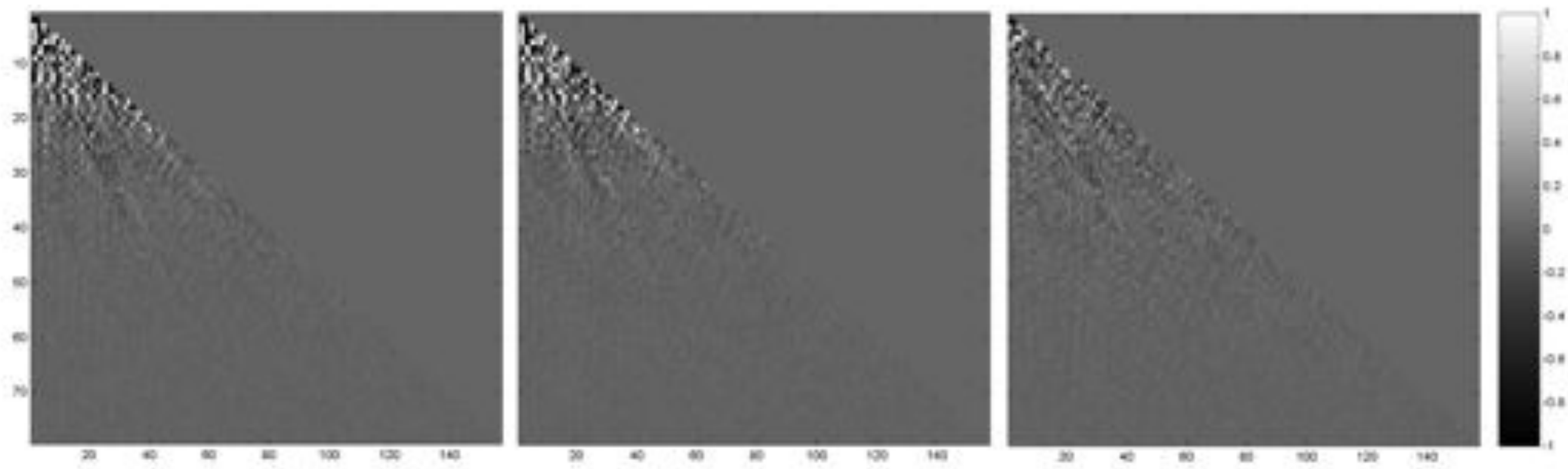
Weighted-SPHARM of cortical thickness



Weighted-SPHARM at different scale



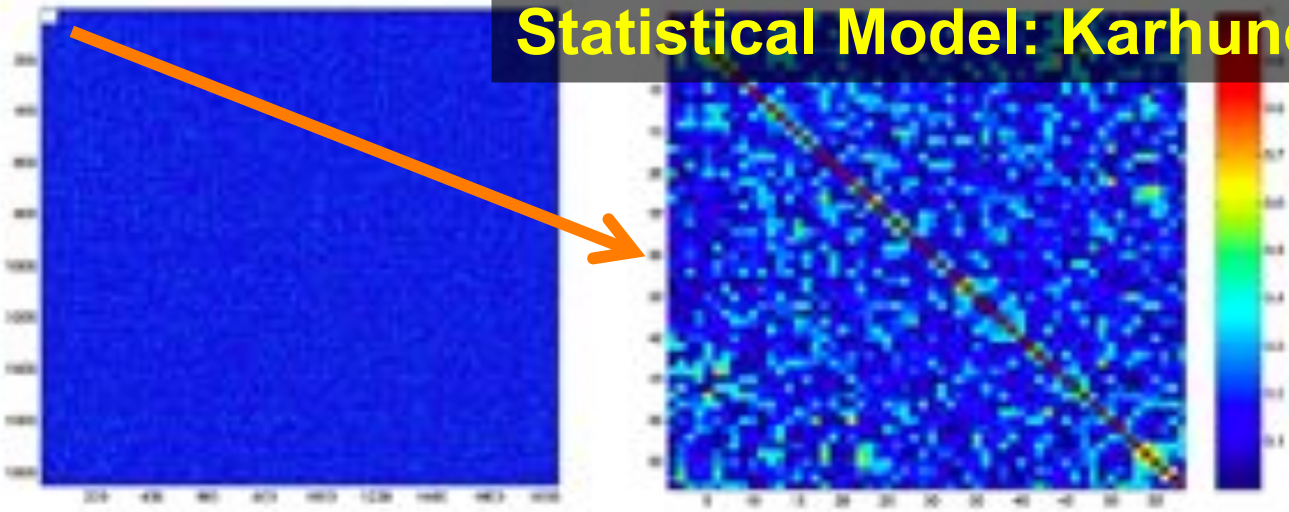
78th degree SPHARM representation



autism **control** **difference**

The coefficients are treated as a multivariate measure and feed into classification techniques.

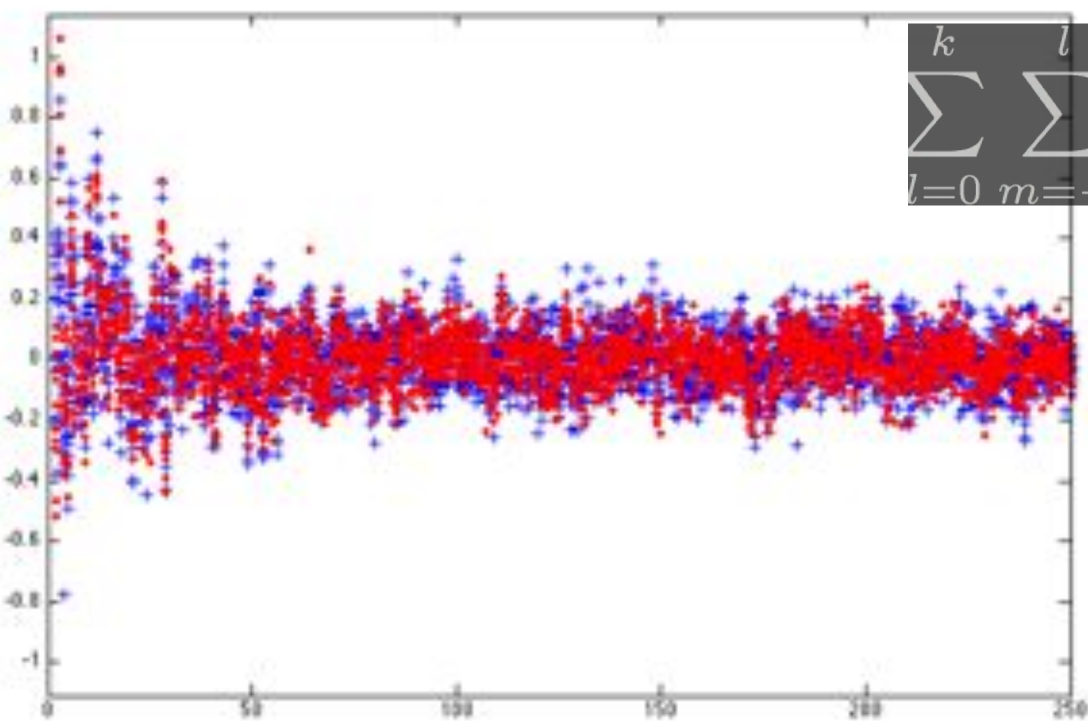
Statistical Model: Karhunen-Loeve expansion



Cross correlations

uncorrelated normal

$$\sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)t} \langle f, Y_{lm} \rangle Y_{lm}(\theta, \varphi)$$



Classification ?
Discrimination?

MATLAB Demonstration

Lecture 4

Iterative linear model fitting methods:

Matching Pursuits

Iterative Residual Fitting (IRF) algorithm

Read

chung.2008.sinica
mallat.1993