Neuroimage Processing

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Lecture 08-09.

Surface-based Morphometry (SBM)

and spherical harmonic representation

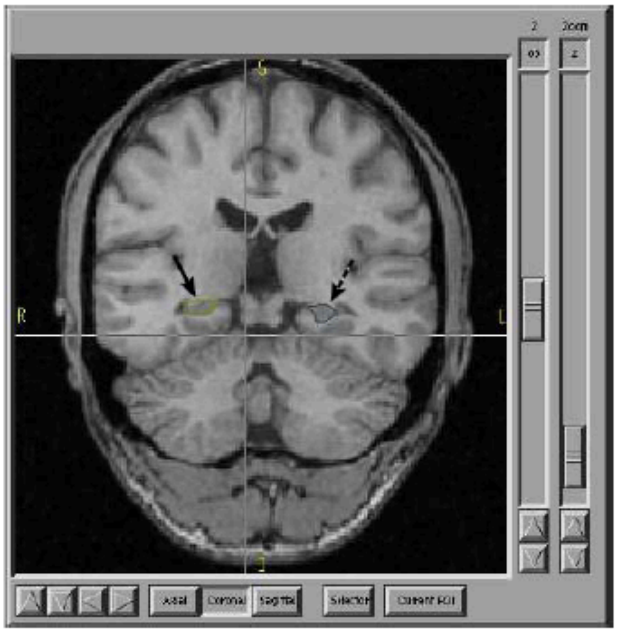
October 30, 2009

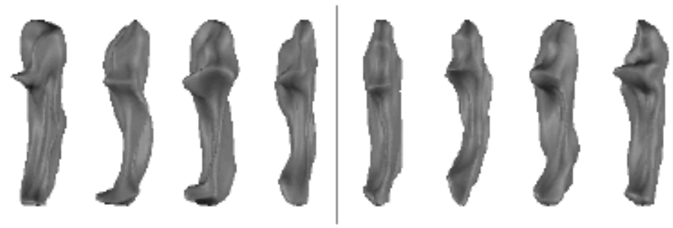
Surface-based Morphometry (SBM)

- Surface-specific morphometric technique.
- More sensitive to surface-specific changes.
- Example: substantial cortical changes were found in AD and other dementias and it is likely that different clinical populations will exhibit different cortical shape variability.

Hippocampal Shape in Schizophrenia

Volume





Surface

Healthy

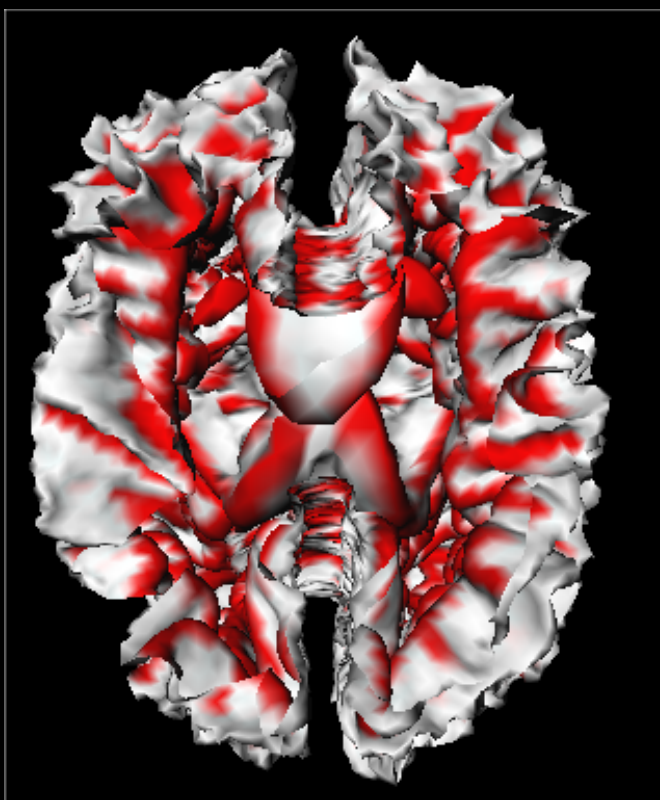
Schizophrenic

Classifier:

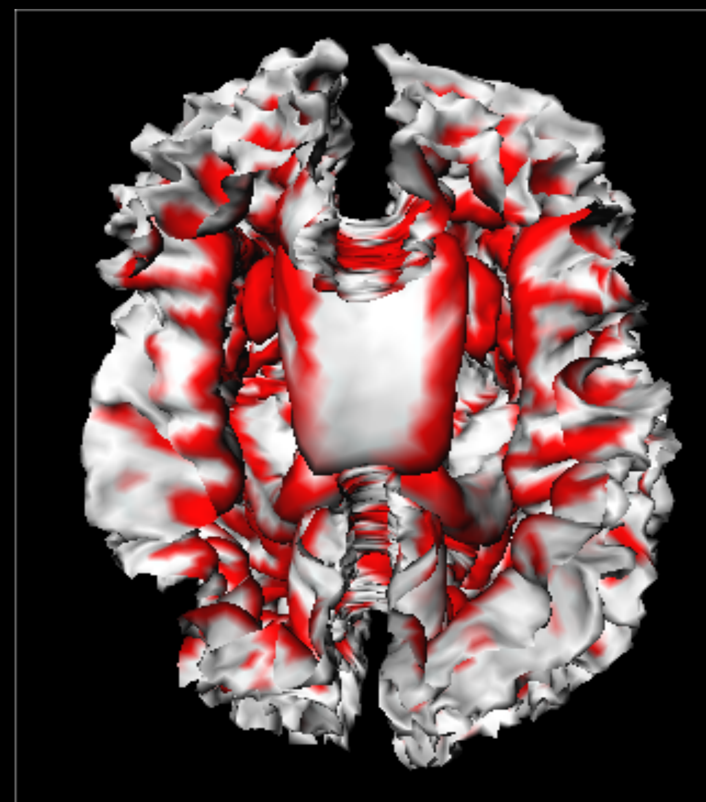
Schizophrenic or not?

Motivation for SBM

Compared to other 3D volumetric techniques, SBM can quantify cortical variations better.



Ventricle enlargement



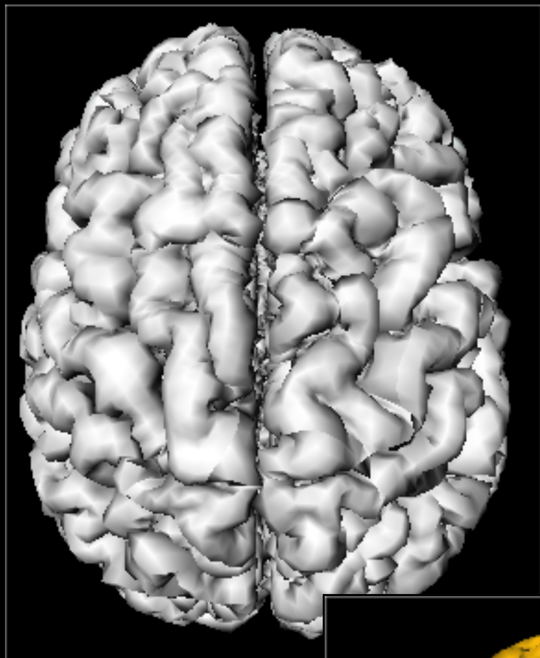
Age 14

Age 19

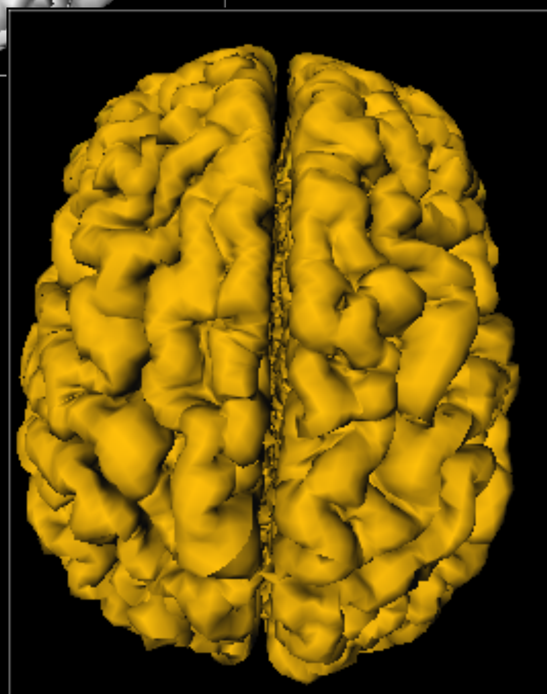
Comparison of cortical surface

Volumetric approach will not be able to detect surface specific change

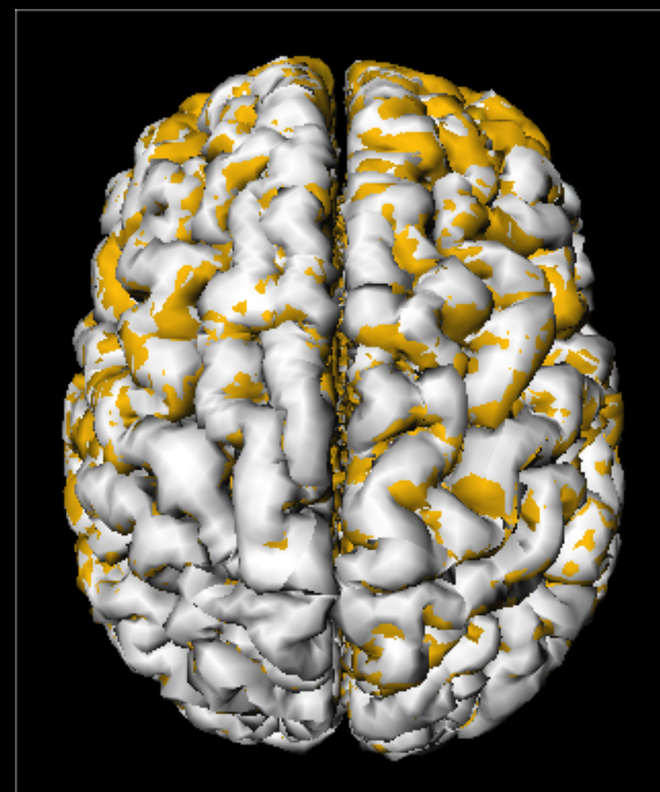
Age 14



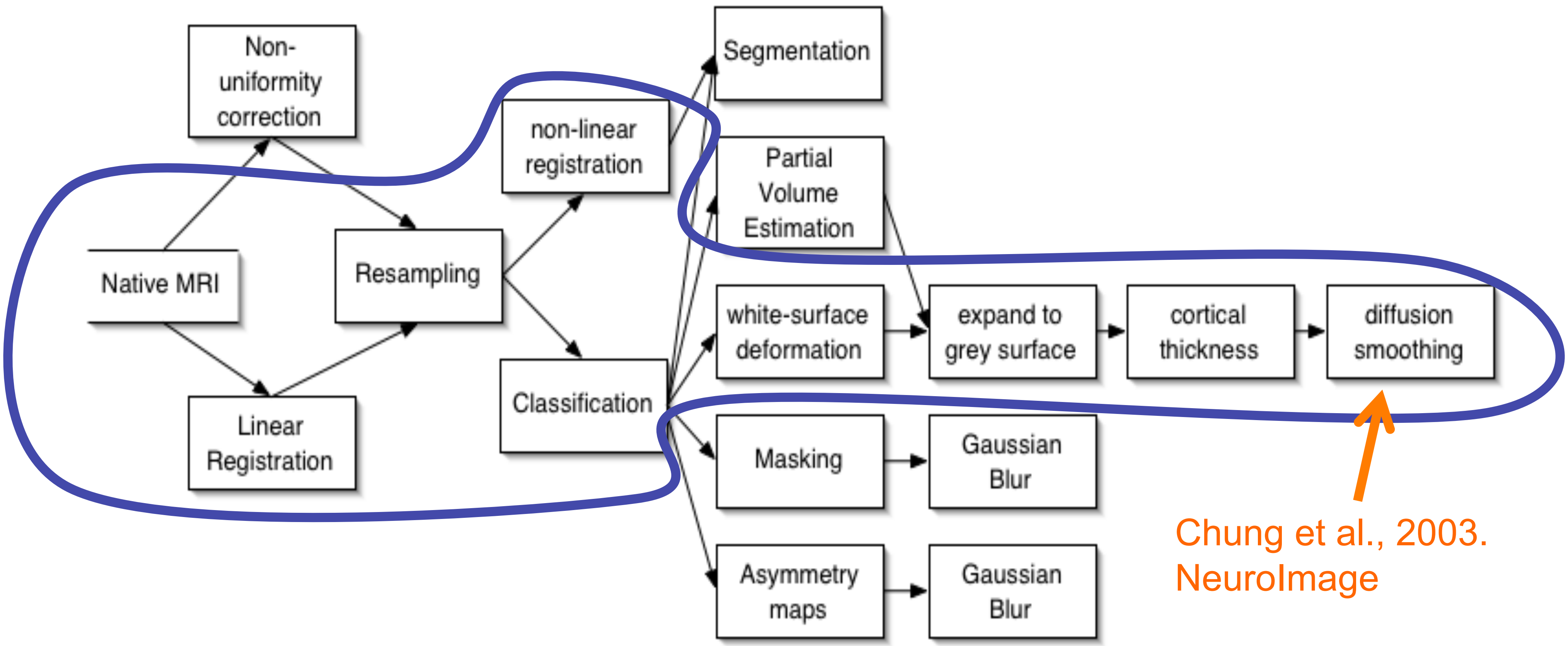
Age 19



Superimposition



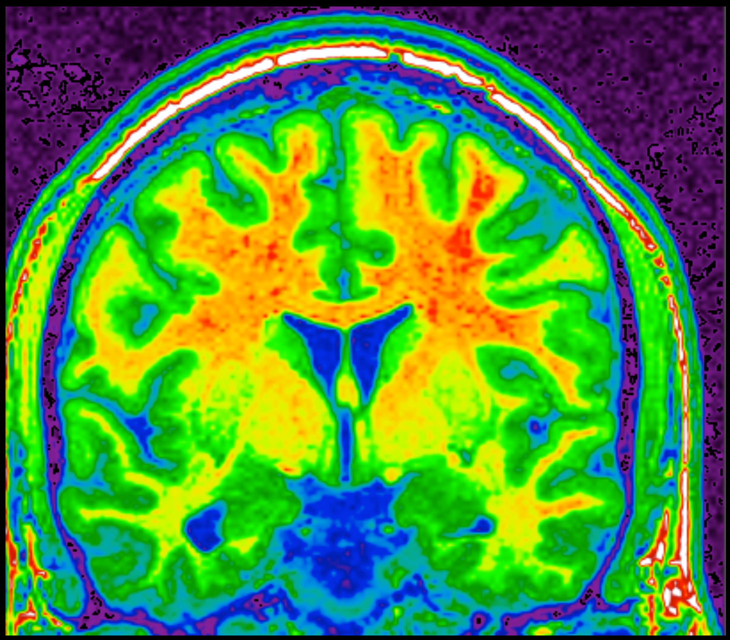
MNI cortical image processing pipeline



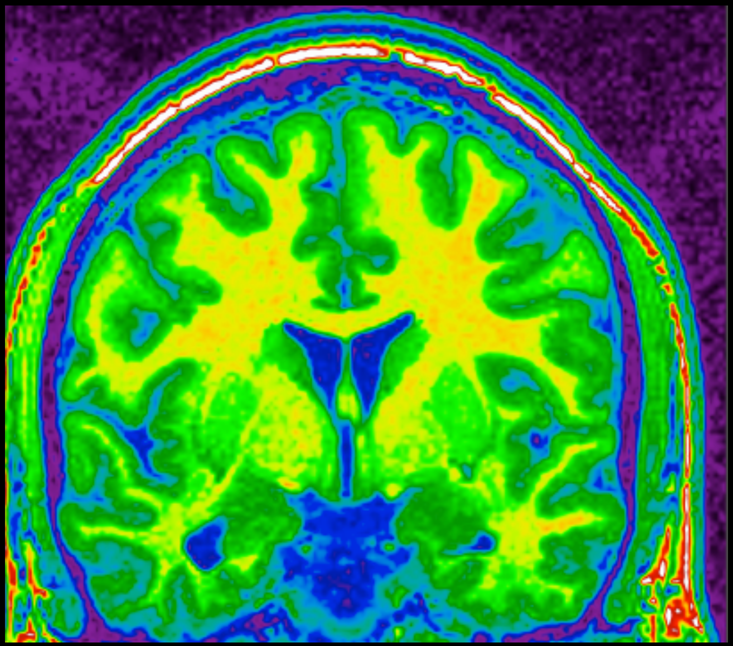
Each component was
developed by a PhD student

Image intensity Nonuniformity correction
N3 algorithm

Original data



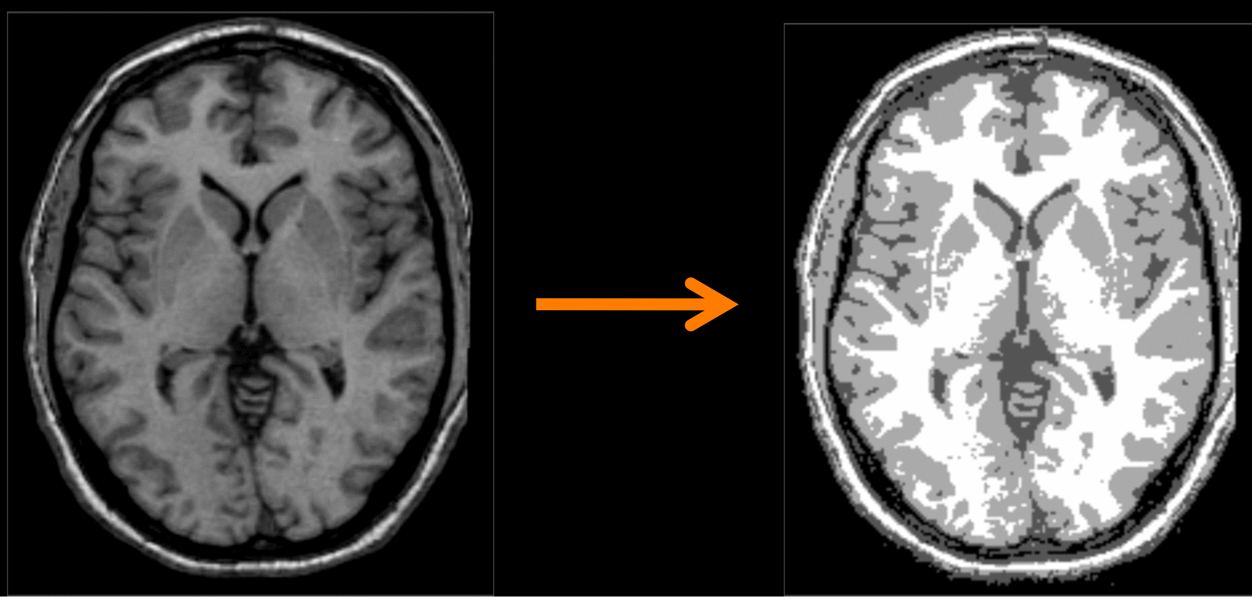
Corrected



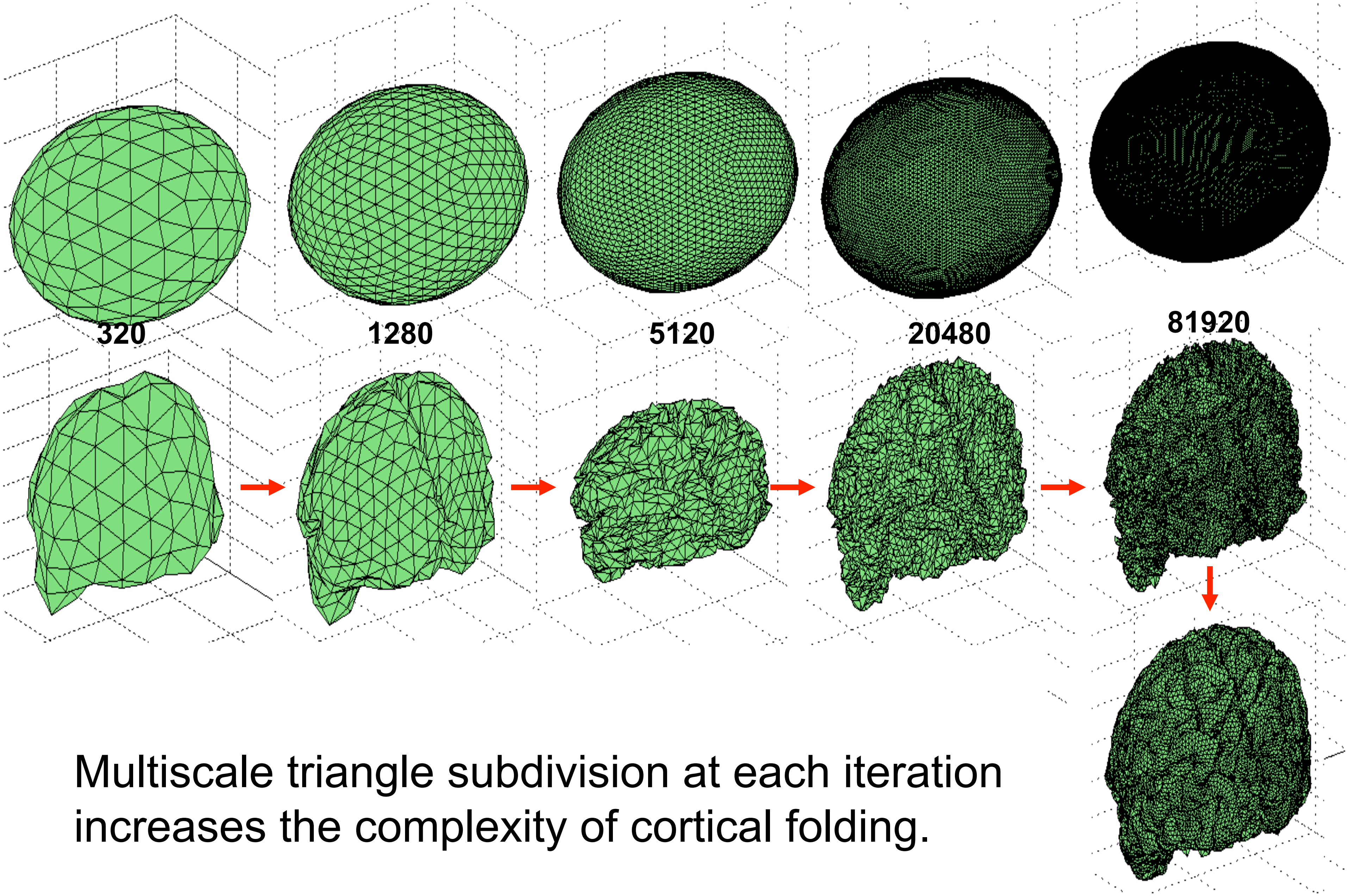
Source: Jason Lerch, MNI

Gaussian mixture modeling

MNI Neural network classifier

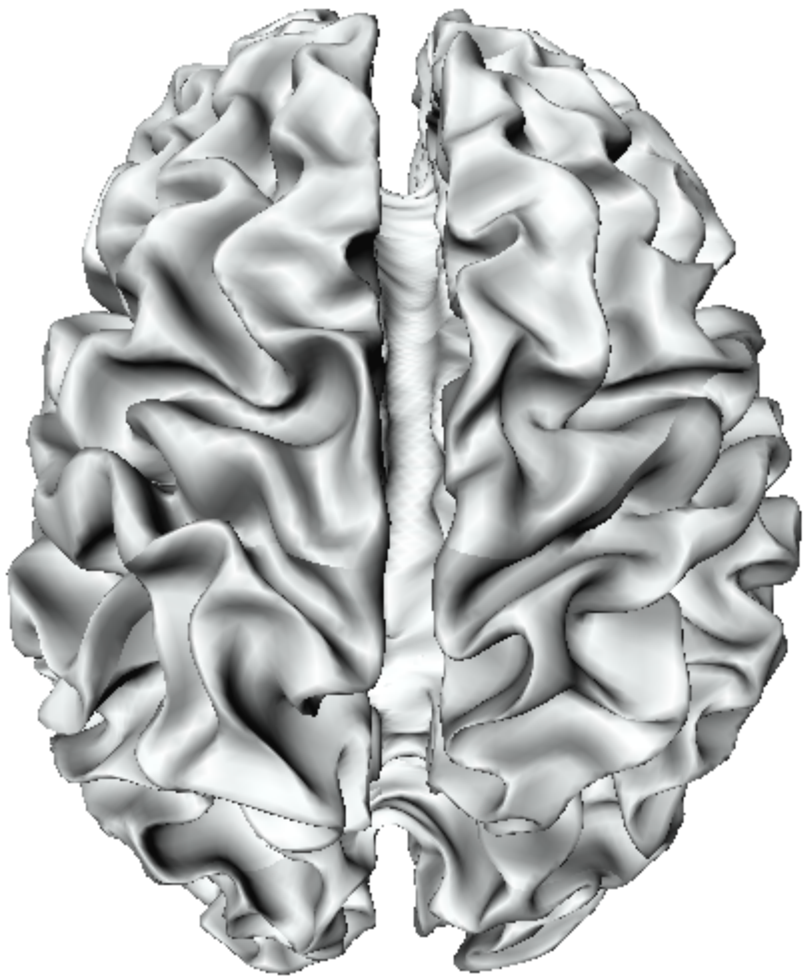


Deformable surface algorithm McDonalds *et al*. (2001) NeuroImage

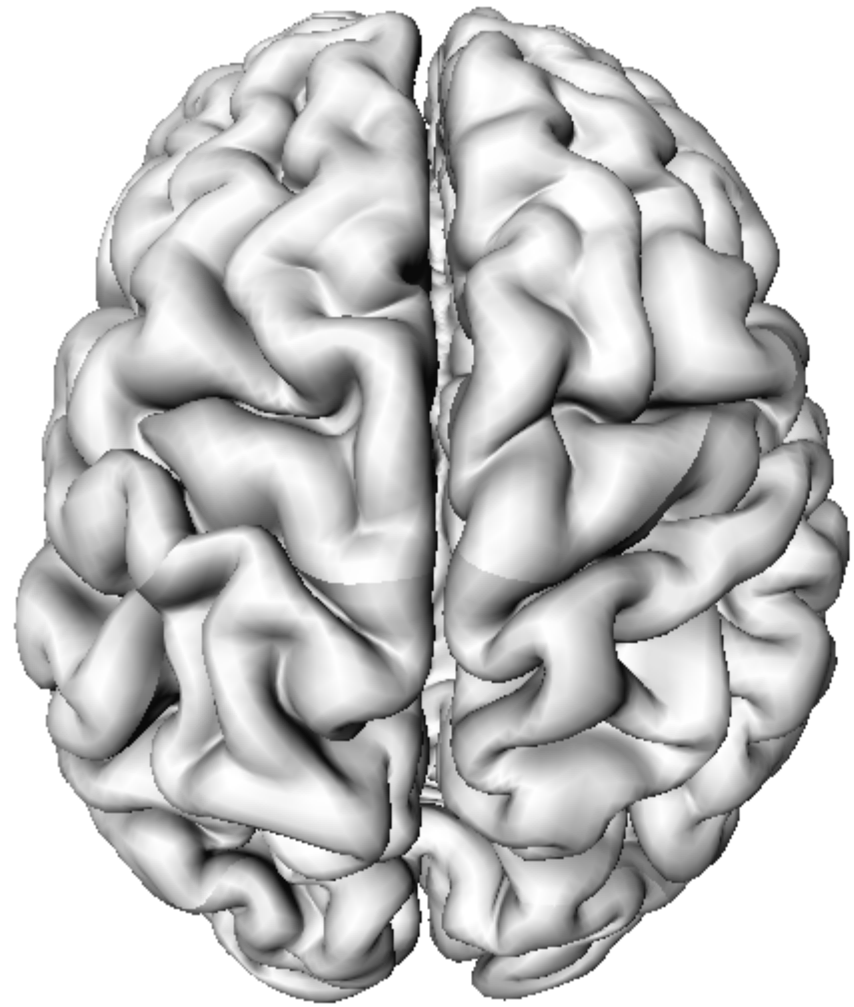


Multiscale triangle subdivision at each iteration increases the complexity of cortical folding.

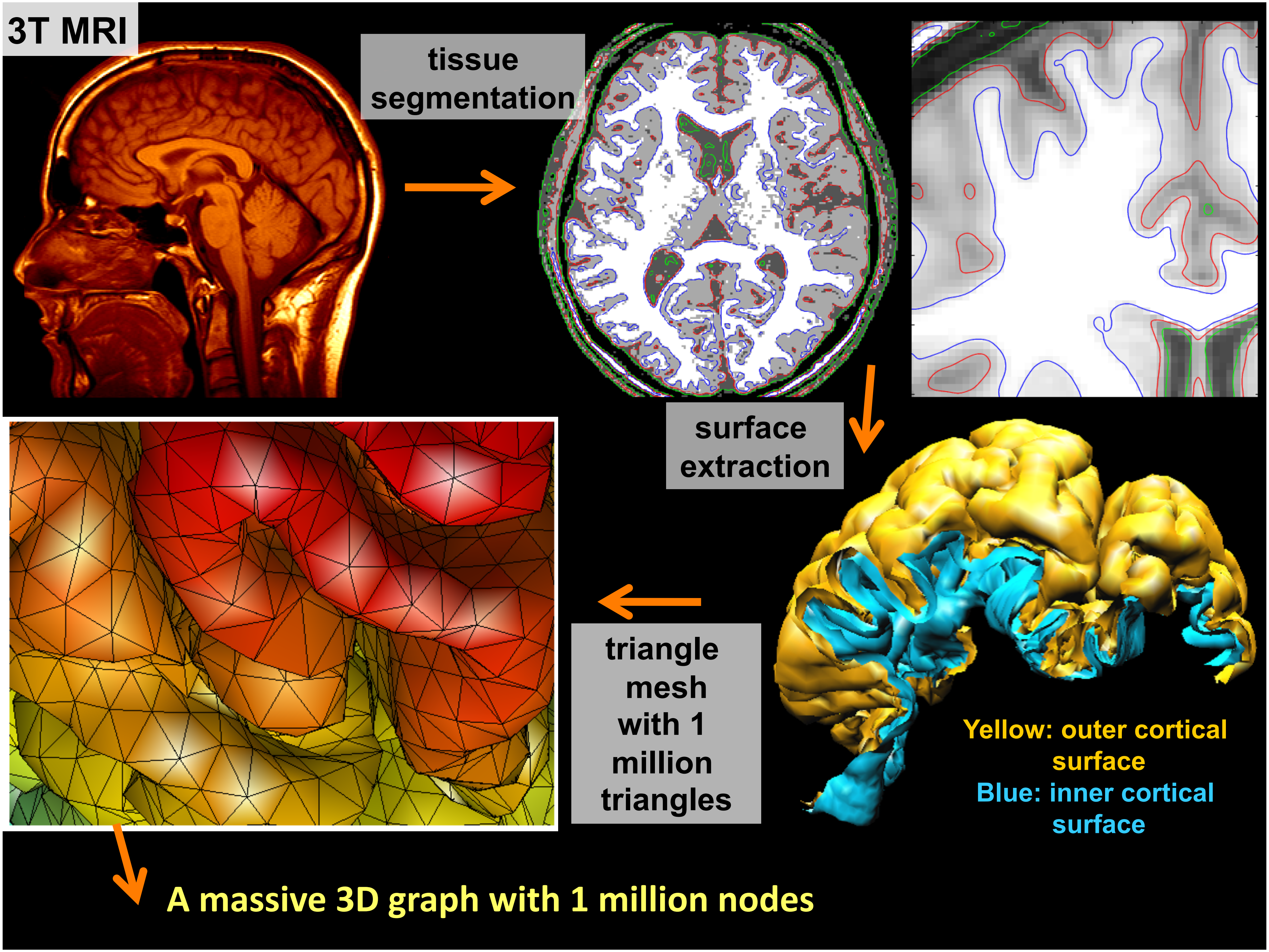
Final surface extraction result



Inner surface

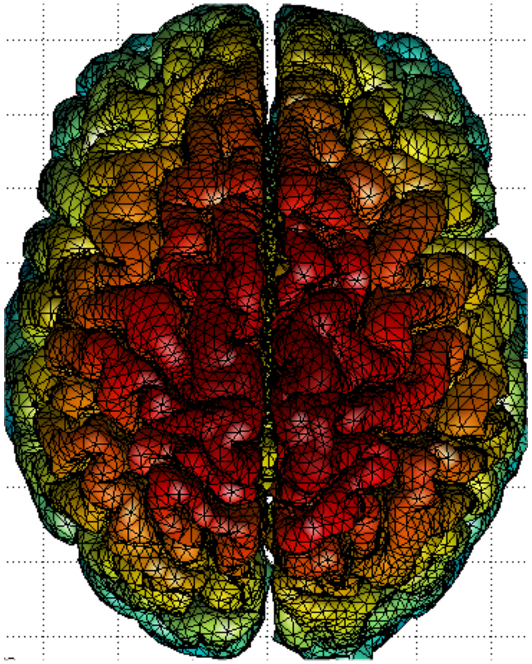


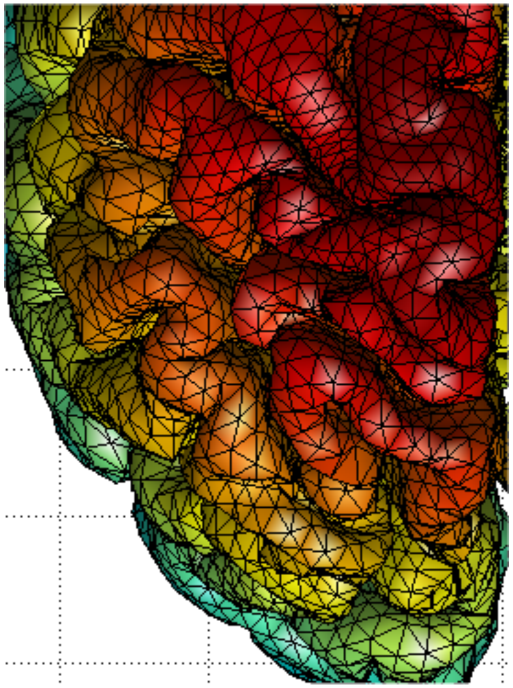
Outer surface

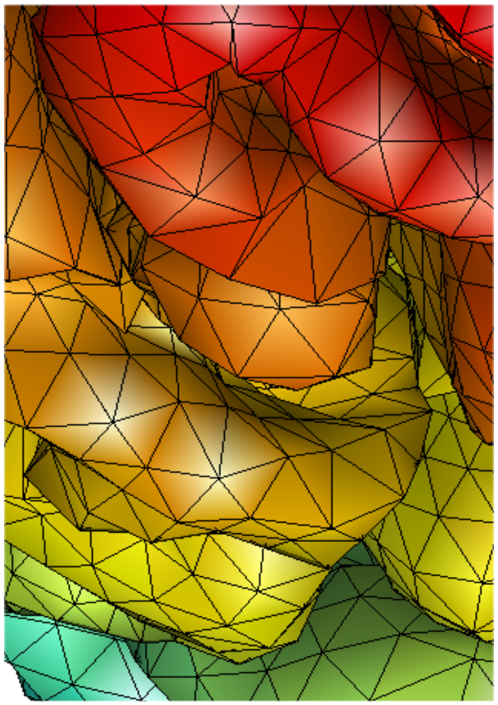

3T MRI
tissue segmentation
surface extraction
triangle mesh with 1 million triangles
Yellow: outer cortical surface
Blue: inner cortical surface
A massive 3D graph with 1 million nodes

Polygonal mesh data structure

Basis of most surface rendering tools for 3D computer games: as 3D Max Studio, Maya







Data structure for polygonal mesh

Coordinates for subject 1

Vertex	1	2	3	4	5	6	….	40962
x	57.1876	41.0450	-53.1115	-38.1080	1.8440	-0.2458		
y	21.6388	-56.3448	29.8912	-65.5394	22.9715	9.4176		
z	2.9667	21.1399	-5.5088	23.6724	21.5146	16.9014		
Thickness	5.0	4.9	3.0	2.1	3.4	4.5		

Coordinates for subject 2

Vertex	1	2	3	4	5	6	….	40962
x	53.4240	41.0552	-61.4073	-43.2099	1.6256	-3.9101		
y	22.5535	-56.7731	20.9221	-65.9948	22.7979	29.7043		
z	7.1866	22.4754	-0.1368	21.3962	20.2838	-10.8959		
Thickness	5.5	3.4	2.7	5.1	3.7	4.5		

Corresponding vertices have approximate anatomical homology.

Two available cortical thickness analysis software

FreeSurfer: Bruce Fischl

http://surfer.nmr.mgh.havard.edu

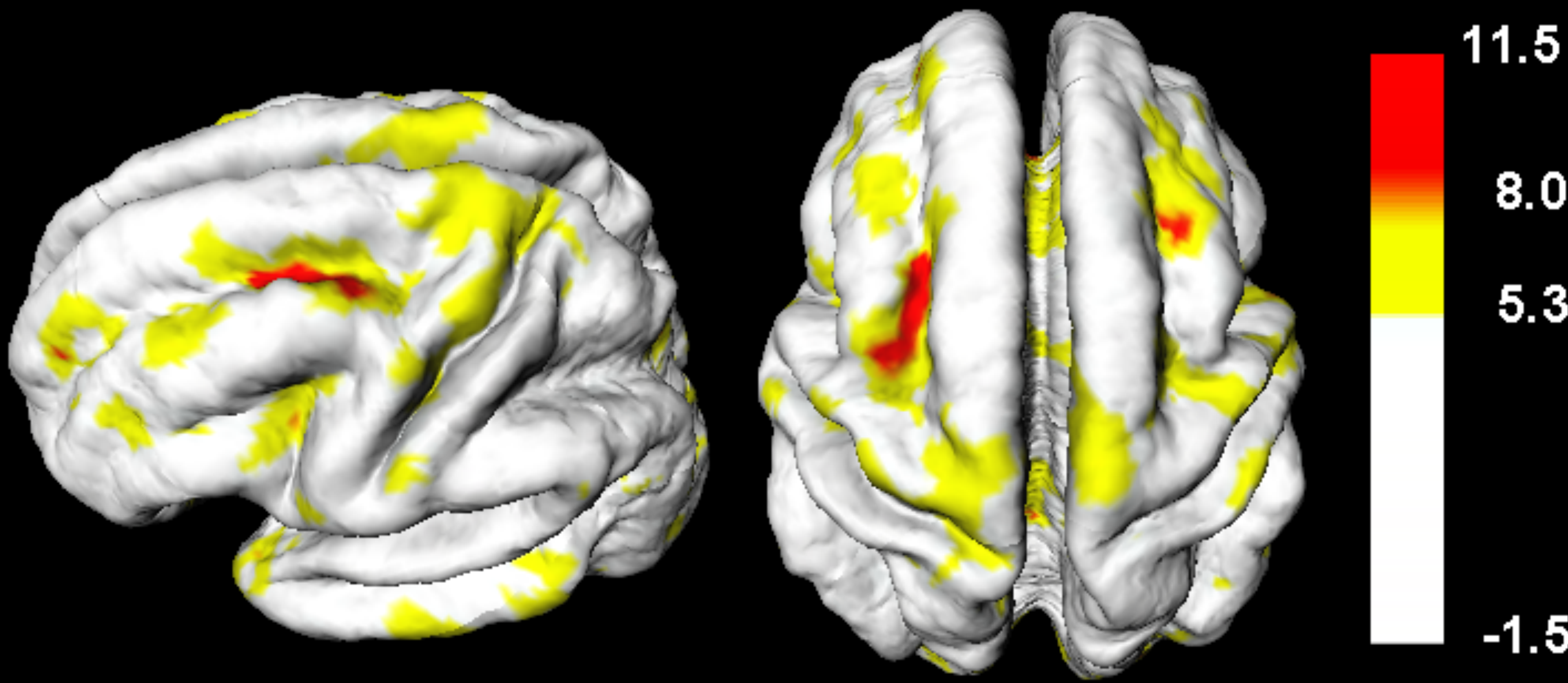
BrainVisa: J.F. Mangin

http://brainvisa.info

Other surface measures

- cortical thickness, curvatures, surface area, tissue density.
- fractal dimension = measure of complexity of anatomical shape.
- Gyrification index

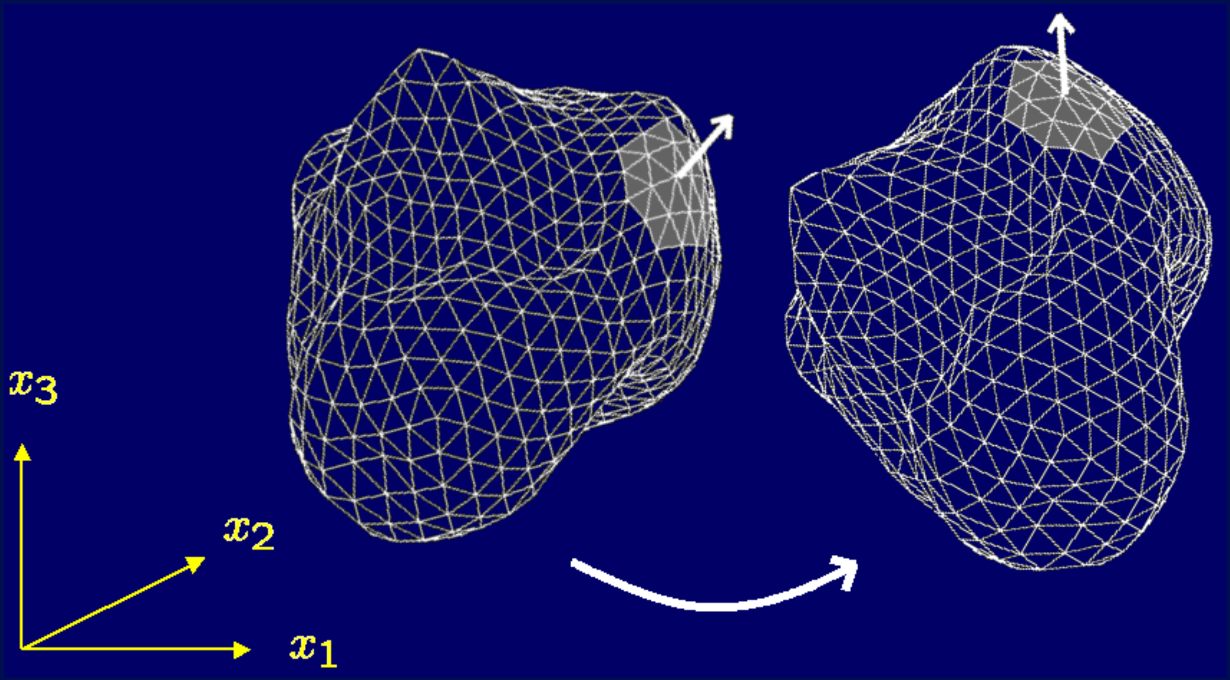
Cortical thickness change *t* map between age 12 and 16.



Curvature estimation via Surface Parameterization

Global: tensor splines, SPHARM
Local: quadratic surface fitting

$$X(u^1, u^2) = \begin{pmatrix} x_1(u^1, u^2) \\ x_2(u^1, u^2) \\ x_3(u^1, u^2) \end{pmatrix}$$

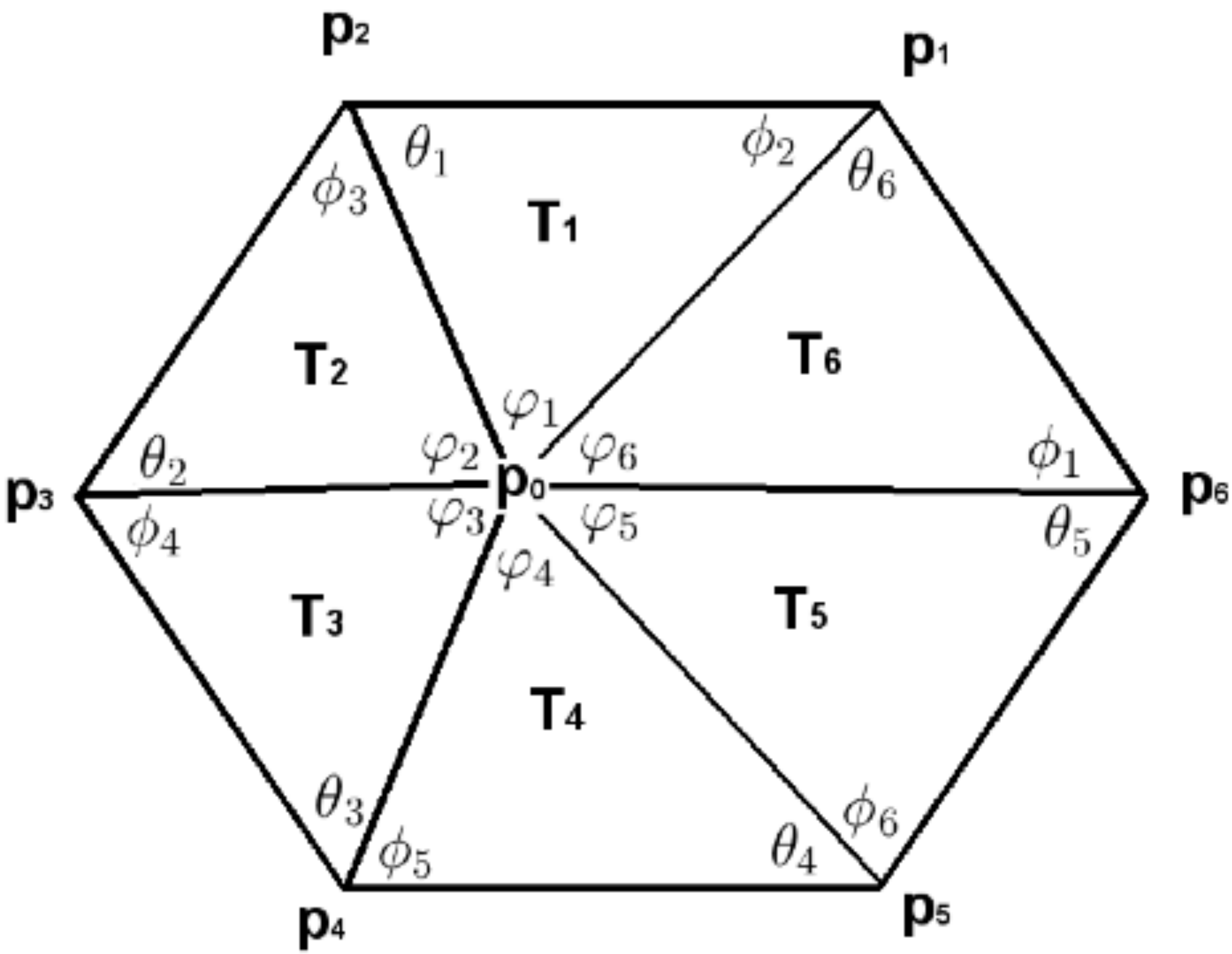


Find the best fitting tangent plane via PCA

Read chapter 6.3 of M.K.Chung.Book.2009.pdf

$$s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots$$

Read chapter 6.3 of M.K.Chung.Book.2009.pdf for detail



If $\mathbf{p}_1, \ldots, \mathbf{p}_m$ are m neighboring points of $\mathbf{p} = \mathbf{p}_0$ in the counter-clockwise direction with respect to the tangent plane $T_{\mathbf{p}}(\partial\Omega)$ at $\mathbf{p}$ (Figure 6.1), then the unit normal vector $\mathbf{n}$ is estimated as

$$\mathbf{n} = \frac{\sum_{i=1}^{m} \varphi_i \mathbf{n}_i}{\sum_{i=1}^{m} \varphi_i},$$

where the unit vectors $\mathbf{n}_i$ are normal to each triangle T_i.

$$\mathbf{n}_i = \frac{(\mathbf{p}_{i+1} - \mathbf{p}) \times (\mathbf{p}_i - \mathbf{p})}{\|(\mathbf{p}_{i+1} - \mathbf{p}) \times (\mathbf{p}_i - \mathbf{p})\|}$$

and the interior angles are

$$\varphi_i = \cos^{-1} \frac{\langle \mathbf{p}_{i+1} - \mathbf{p}, \mathbf{p}_i - \mathbf{p} \rangle}{\|\mathbf{p}_{i+1} - \mathbf{p}\| \|\mathbf{p}_i - \mathbf{p}\|}.$$

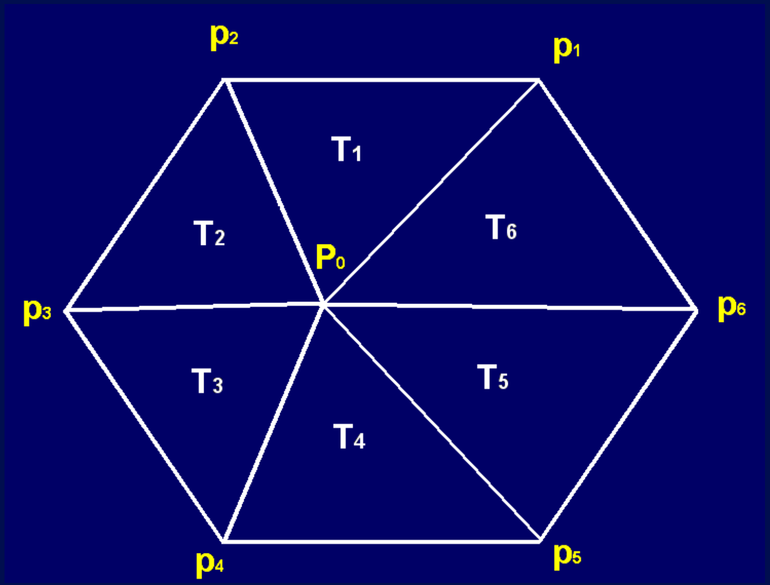
Alternatively, we may employ a method similar to principal components analysis (PCA). The equation of the plane with the unit normal vector $\mathbf{n}$ passing through the point p is $\langle \mathbf{n}, x \rangle = \langle \mathbf{n}, \mathbf{p} \rangle$. The distance from the point $\mathbf{p}_i$ to the plane is the length of the projection of $\mathbf{p}_i - \mathbf{p}$ onto the unit normal vector $\mathbf{n}$, i.e. $\langle \mathbf{n}, \mathbf{p}_i - \mathbf{p} \rangle$. Then we find the best fitting tangent plane in the sense of minimizing the sum of squared distance of the points $\mathbf{p}_1, \ldots, \mathbf{p}_m$ to the plane:

$$\min_{\mathbf{n}} \sum_{i=1}^{m} \langle \mathbf{n}, \mathbf{p}_i - \mathbf{p} \rangle^2 = \min_{\mathbf{n}} \ \mathbf{n}^t \mathbf{C} \mathbf{n},$$

where $\mathbf{C} = \sum_{i=1}^{m} (\mathbf{p}_i - \mathbf{p})(\mathbf{p}_i - \mathbf{p})^t$. If the fitting plane is not forced to pass through the point p, $\mathbf{C}$ becomes the sample covariance matrix of $\mathbf{p}_1, \ldots, \mathbf{p}_m$ and the optimization problem is exactly the standard PCA. Since $\mathbf{n}^t\mathbf{n} = 1$, using the Lagrange multiplier γ minimize $\mathbf{n}'\mathbf{C}\mathbf{n} - \gamma(\mathbf{n}'\mathbf{n} - 1)$. Differentiating with respect to $\mathbf{n}$, $\mathbf{C}\mathbf{n} - \gamma\mathbf{n} = 0$. Thus, γ is an eigenvalue of $\mathbf{C}$. Note that we are minimizing $\mathbf{n}'\mathbf{C}\mathbf{n} = \mathbf{n}'\gamma\mathbf{n} = \gamma$. So the unit normal vector $\mathbf{n}$ of the best fitting tangent plane should be the eigenvector $\mathbf{n}$ that corresponds to the smallest eigenvalue.

MATLAB Demonstration

Polynomial surface fitting
(polynomial regression)
on irregular triangular mesh



$$Y = \mathbb{X}\beta$$

$$\begin{pmatrix} u_1^3 \\ u_2^3 \\ \vdots \\ u_m^3 \end{pmatrix} = \begin{pmatrix} u_1^1 & u_1^2 & (u_1^1)^2 & u_1^1 u_1^2 & (u_1^2)^2 \\ u_2^1 & u_2^2 & (u_2^1)^2 & u_2^1 u_2^2 & (u_2^2)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_m^1 & u_m^2 & (u_m^1)^2 & u_m^1 u_m^2 & (u_m^2)^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

$$\hat{\beta} = (\mathbb{X}'\mathbb{X})^{-}\mathbb{X}'Y$$

Riemannian Metric Tensors

The first fundamental form

$$g_{ij} = \langle \frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j} \rangle$$

The second fundamental form

$$l_{ij} = \langle \mathbf{n}, \frac{\partial^2 X}{\partial u^i \partial u^j} \rangle$$

Mean curvature

$$K_M = \mathrm{tr}(g^{-1} l)/2$$

Gaussian curvature

$$K_G = \det(g^{-1} l) = \det(l)/\det(g)$$

Laplace-Beltrami operator

$$\boxed{\Delta_X F = \frac{1}{|g|^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u^i} \left(|g|^{1/2} g^{ij} \frac{\partial F}{\partial u^j} \right)}$$

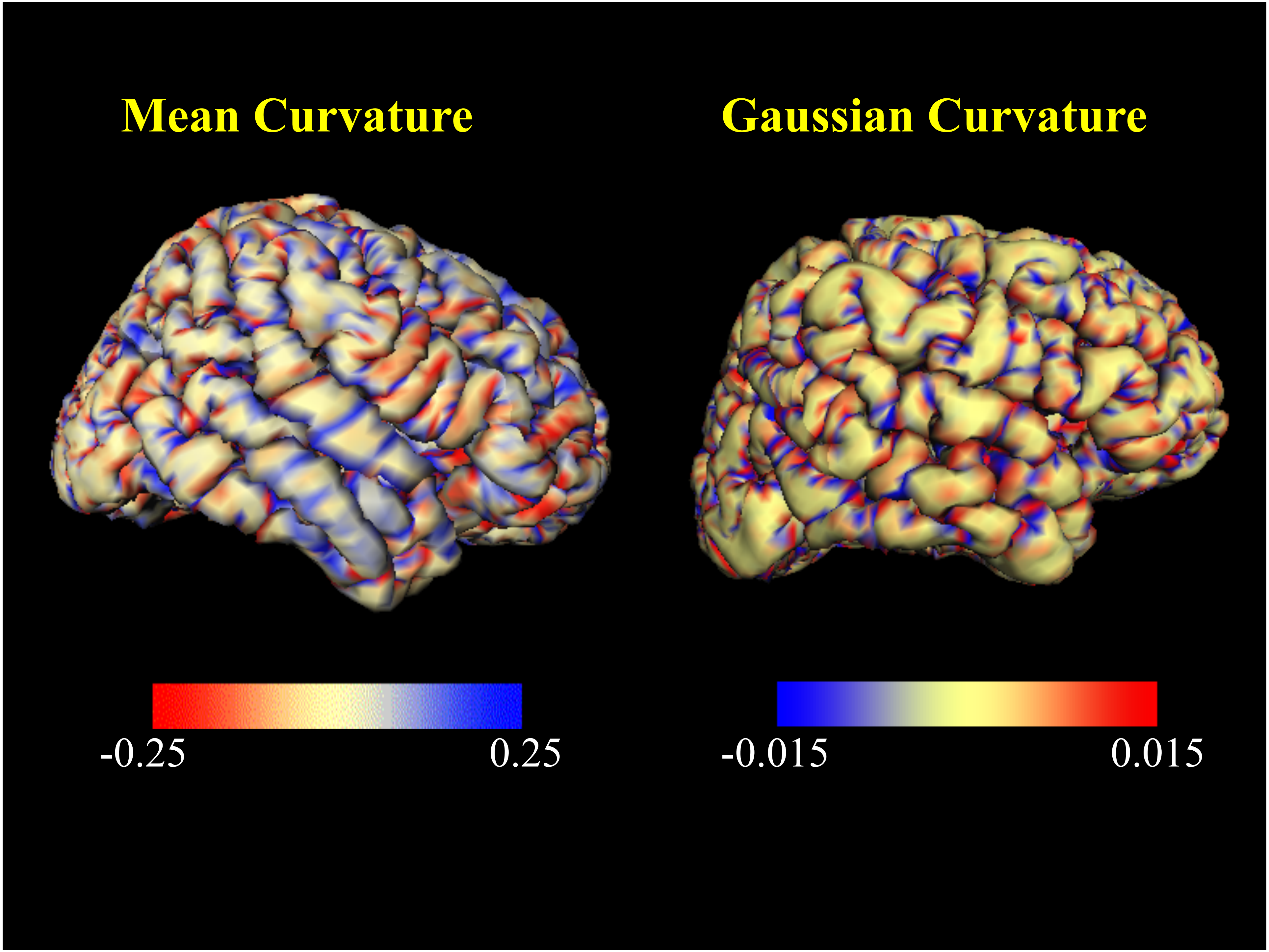
$$s(u^1, u^2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_1^2 + 2\beta_4 u_1 u_2 + \beta_5 u_2^2 + \cdots$$

Why quadratic surface ?

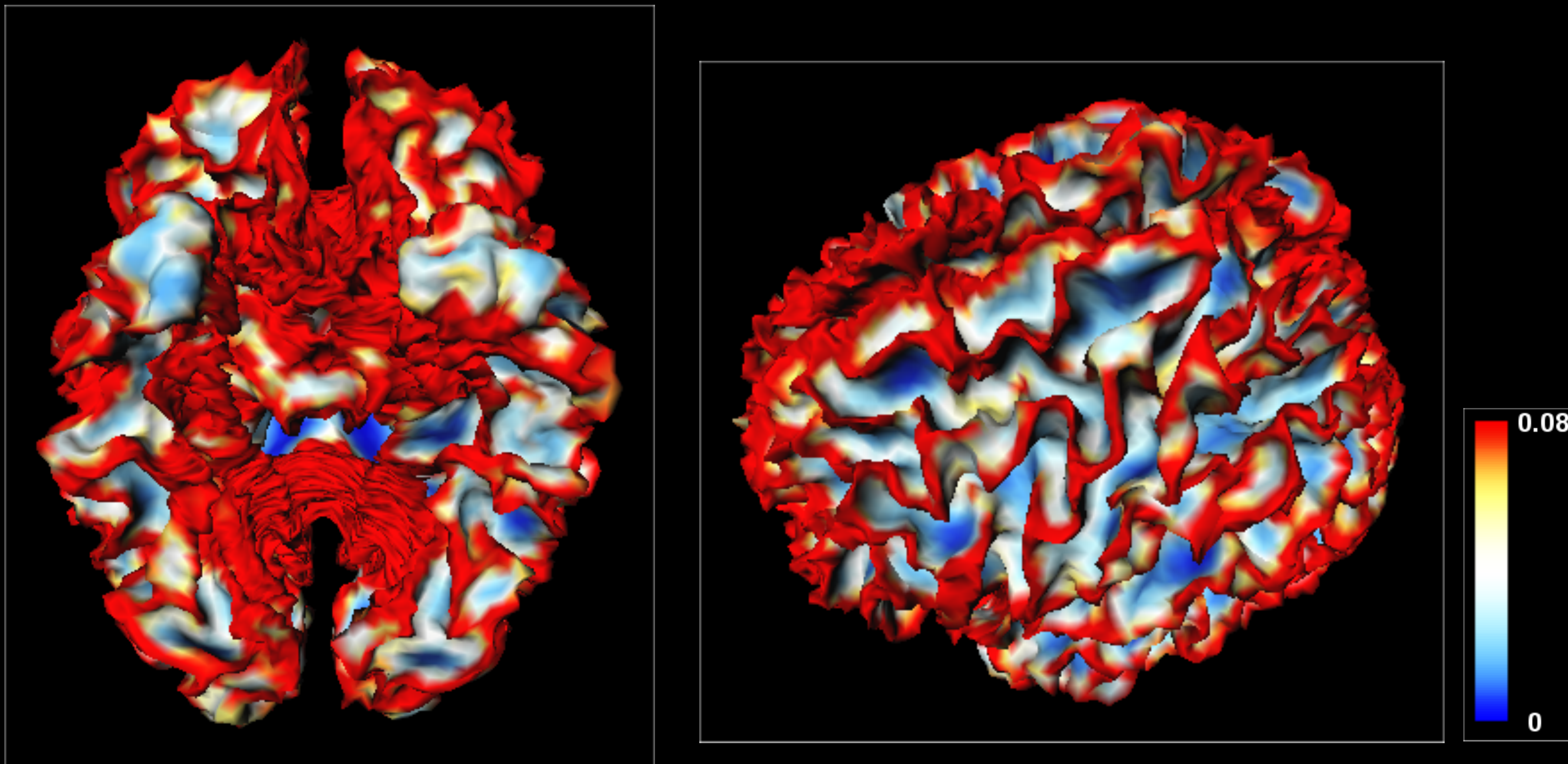
$$g = \begin{pmatrix} 1+\beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & 1+\beta_2^2 \end{pmatrix}$$

$$l = \begin{pmatrix} \beta_3 & \beta_4 \\ \beta_4 & \beta_5 \end{pmatrix}$$

$$K_M = \frac{\mathrm{tr}(g^{-1} l)}{2} = \frac{\beta_3(1+\beta_2^2) + \beta_5(1+\beta_1^2) - 2\beta_1\beta_2\beta_4}{2 + 4(\beta_1^2 + \beta_2^2)}$$


Mean Curvature
Gaussian Curvature
-0.25
0.25
-0.015
0.015

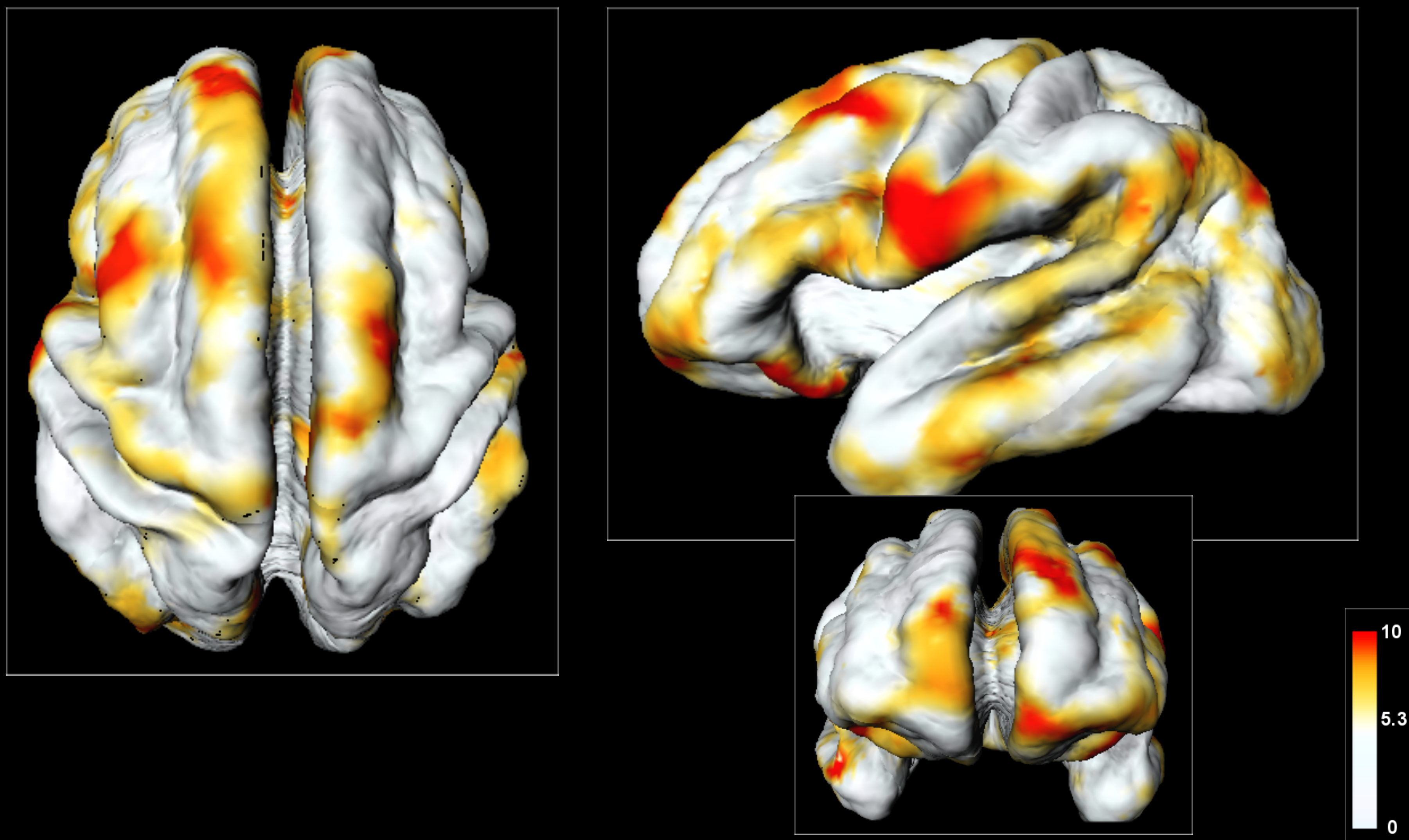
Bending Energy for 14 year old subject



Bending energy or thin-plate spline energy can be used to measure the curvature of the surface.
Between ages 12 and 16, it increases both locally and globally.

CVPR 2003

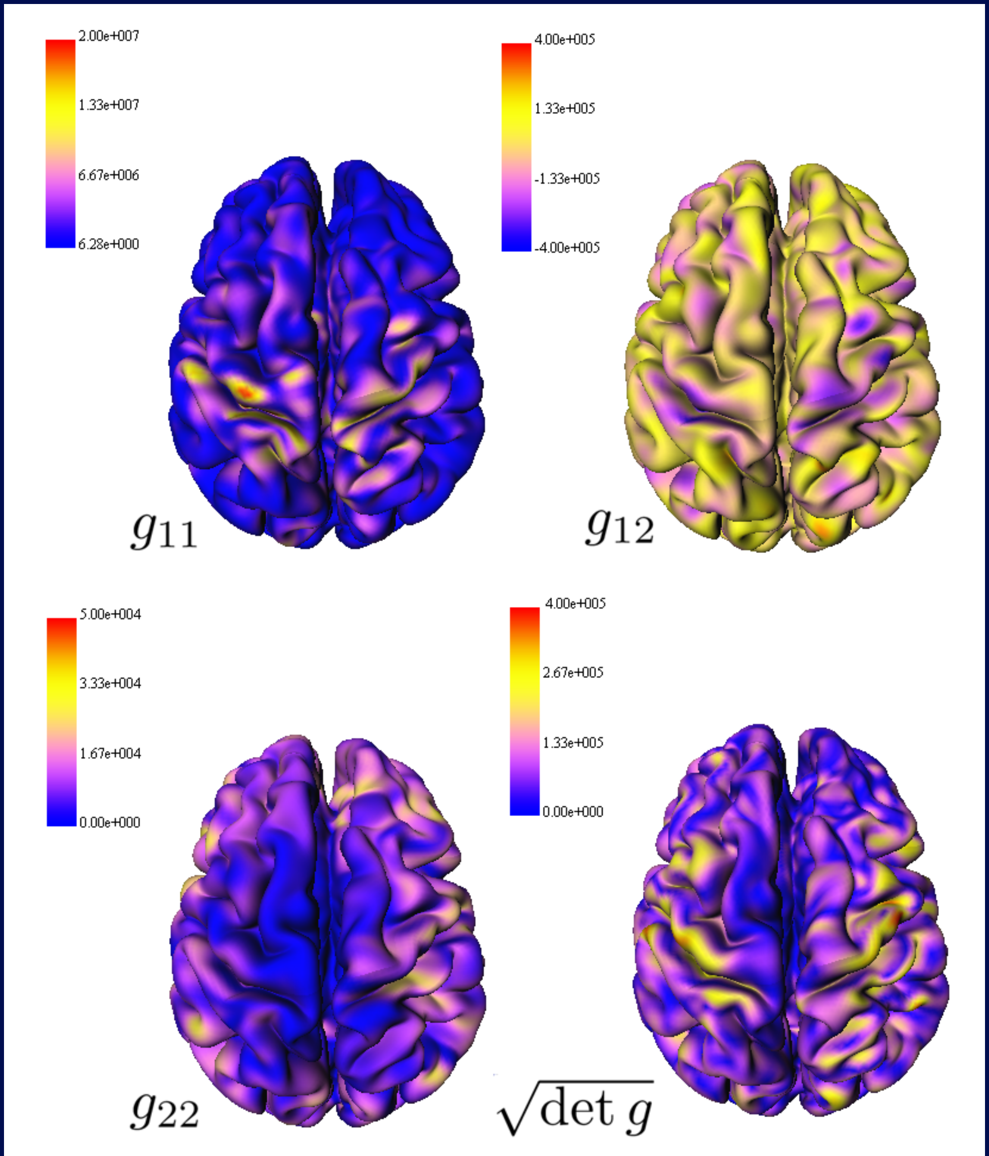
Curvature change *t* map between age 12 and 16



Surface area expansion/shrinking

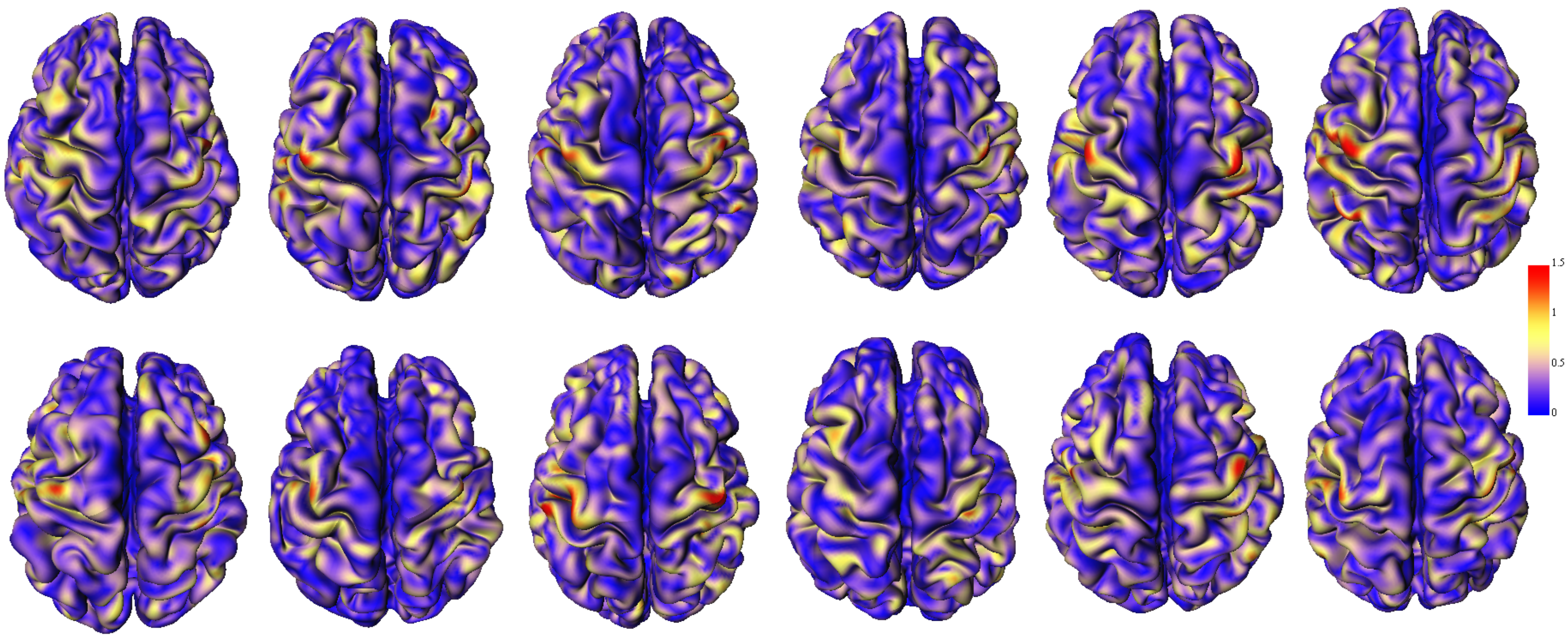
Local surface area element:

$$\sqrt{|g|} = \sqrt{1 + \beta_1^2 + \beta_2^2}$$

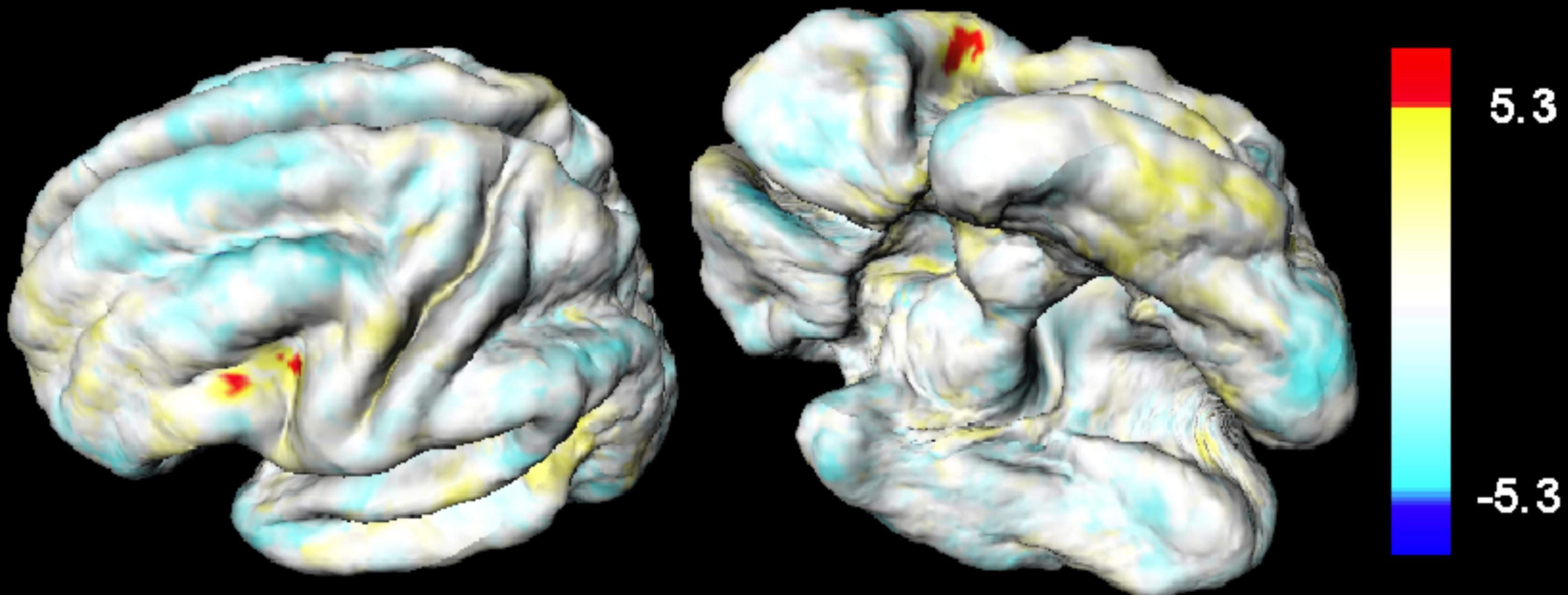


Spherical harmonic representation was used to analytically compute and smooth surface area element

Local area expansion with respect to a template (it ranges between 0 and 1.3)



Surface area change *t* map

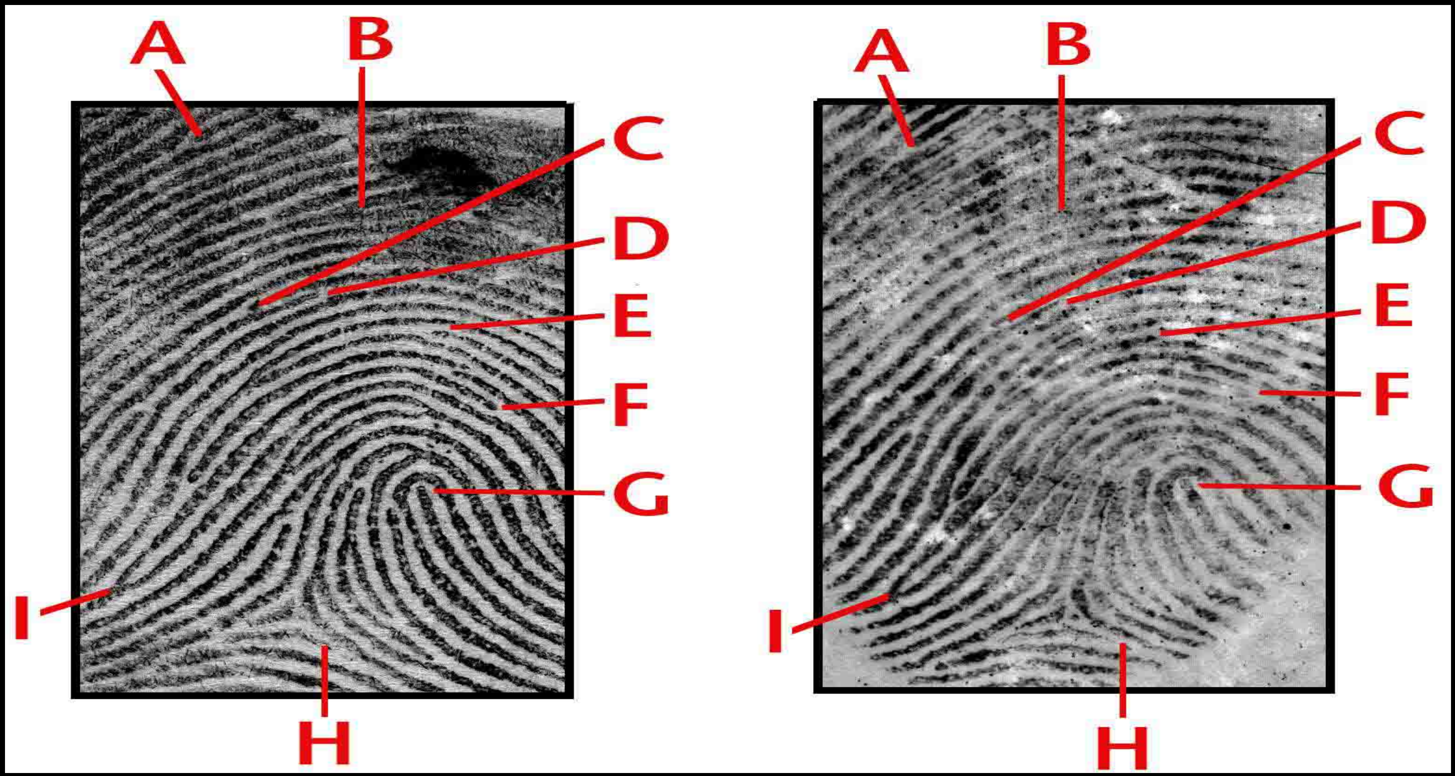


dilatation rate between age 12 and 16

min = - 57 %   mean = - 0.02 %   max = 65 %

Surface Registration

In order to compare cortical measures across subjects, it Is necessary to find a mapping between homologous anatomical regions.

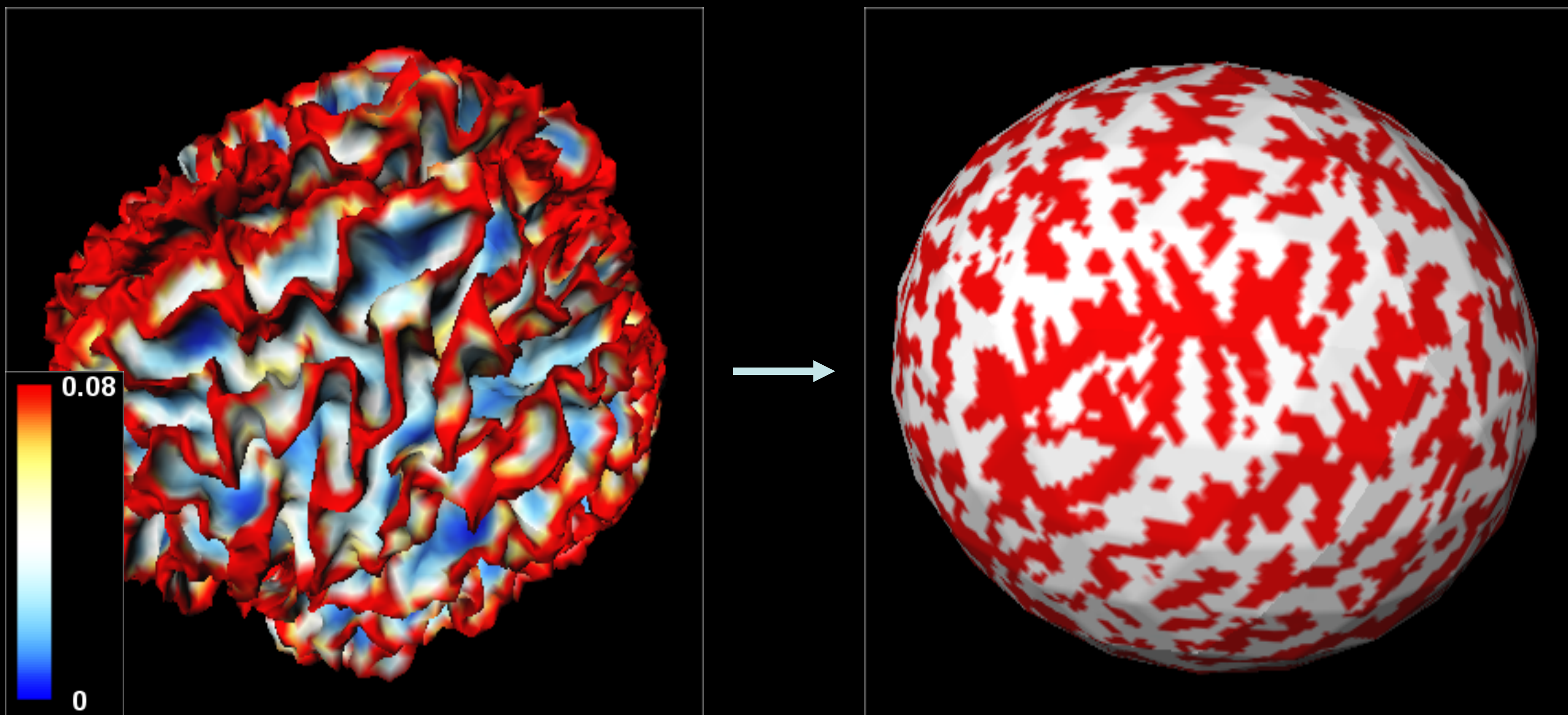


Sulcal pattern alignment= finger print of brain

Compute cortical curvature and map curvature to unit sphere

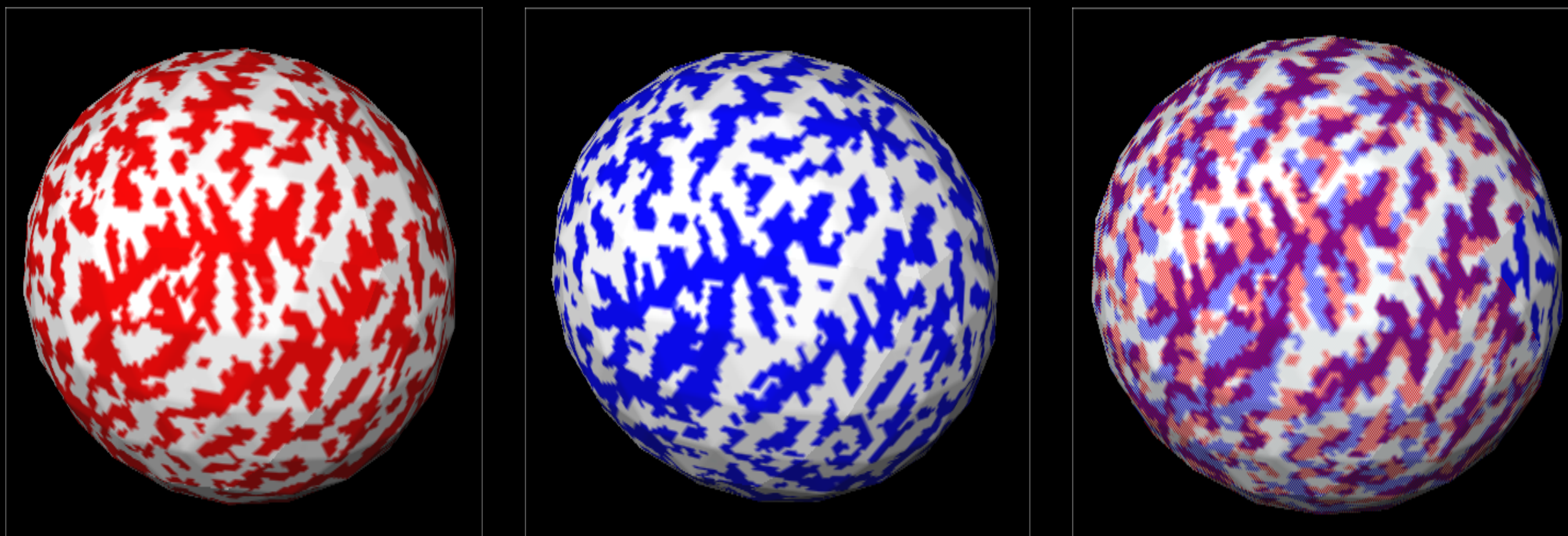
3D problem

2D problem



Unit sphere gives a natural coordinate system
(spherical coordinates).

Sulcal pattern matching



Misalignment

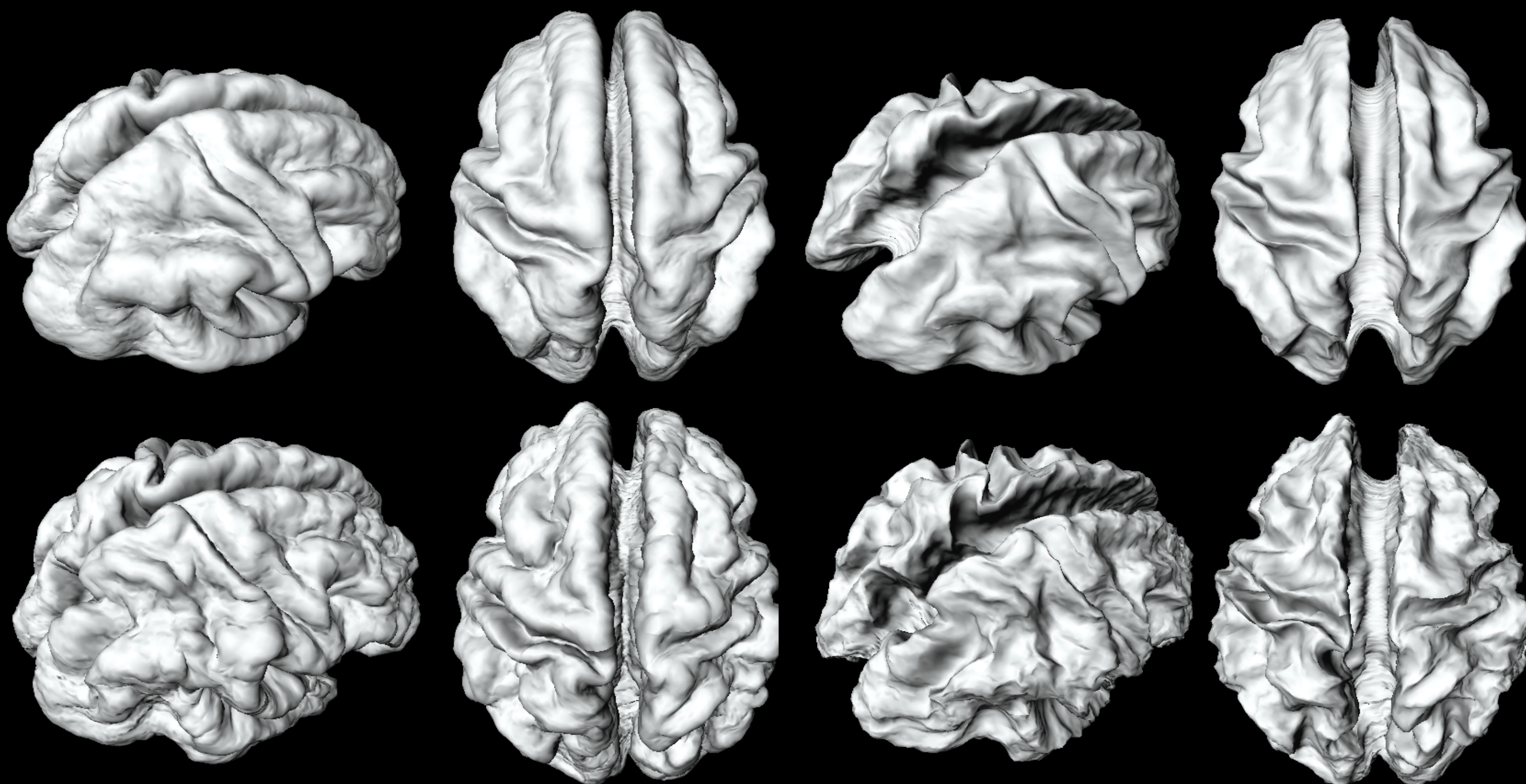
Sulcal pattern matching by minimizing

objective function = curvature difference - smoothness of deformation

See Paul Thompson's earlier IEEE TMI paper

Average Template construction

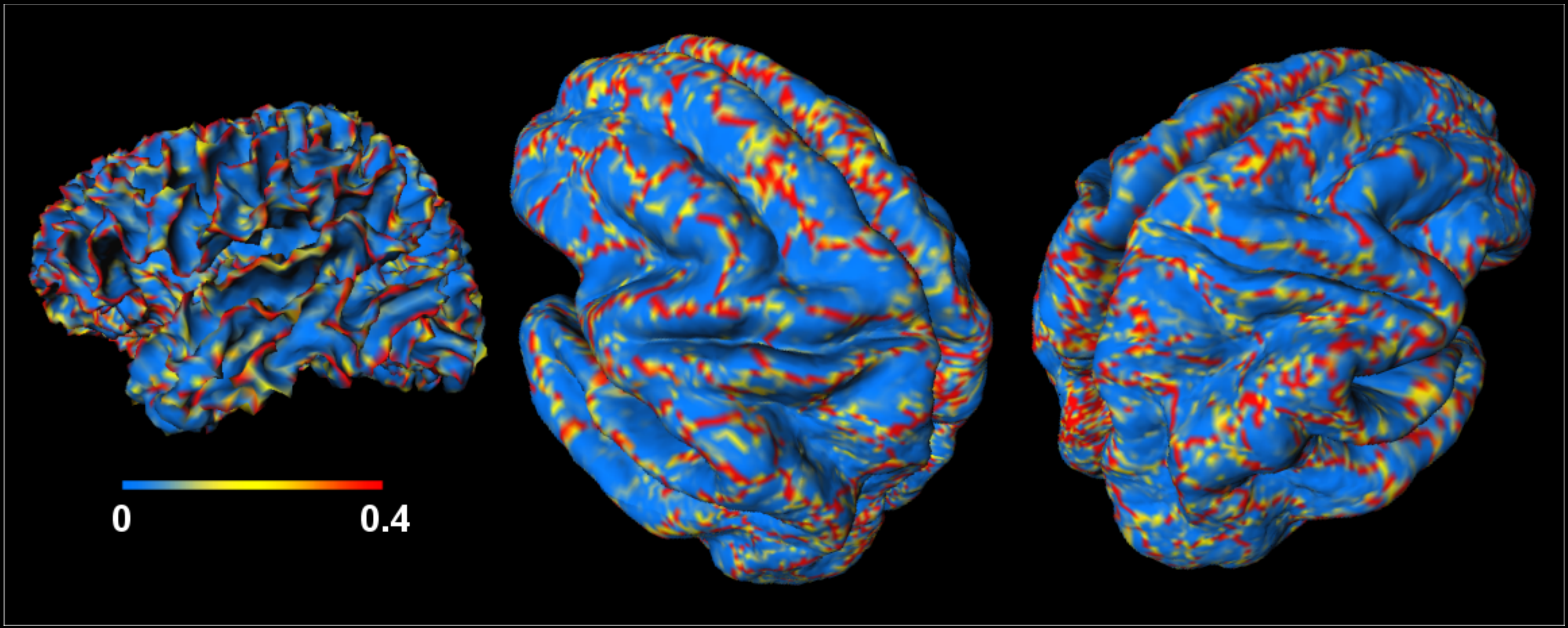
Averaged corresponding coordinates of individual surface



Top: Old technique Chung et al. NeuroImage (2003)

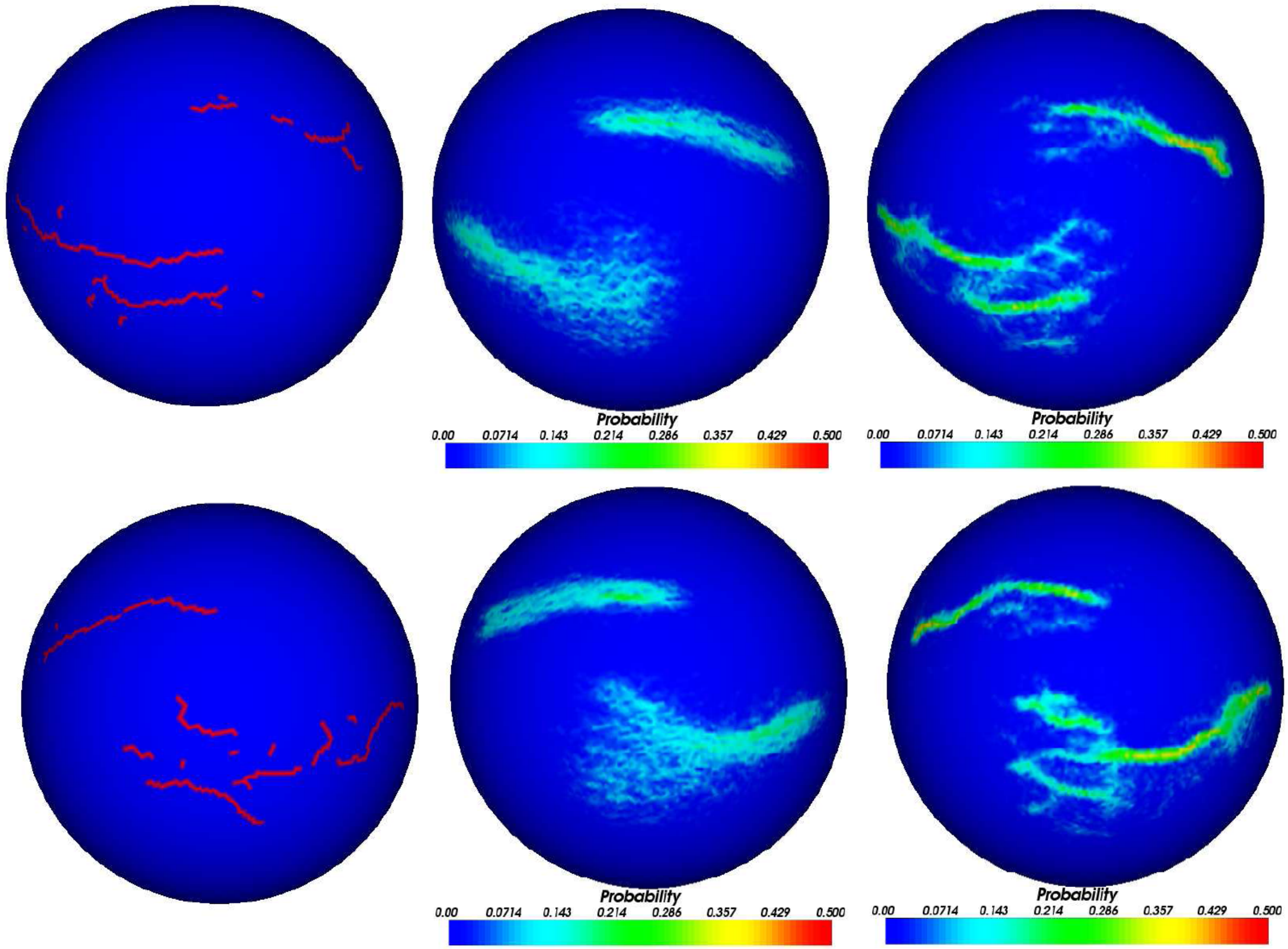
Bottom: New technique Chung et al. NeuroImage (2005)

Demonstrating the alignment of sulci on average template



Principle curvature maps projected on the average template

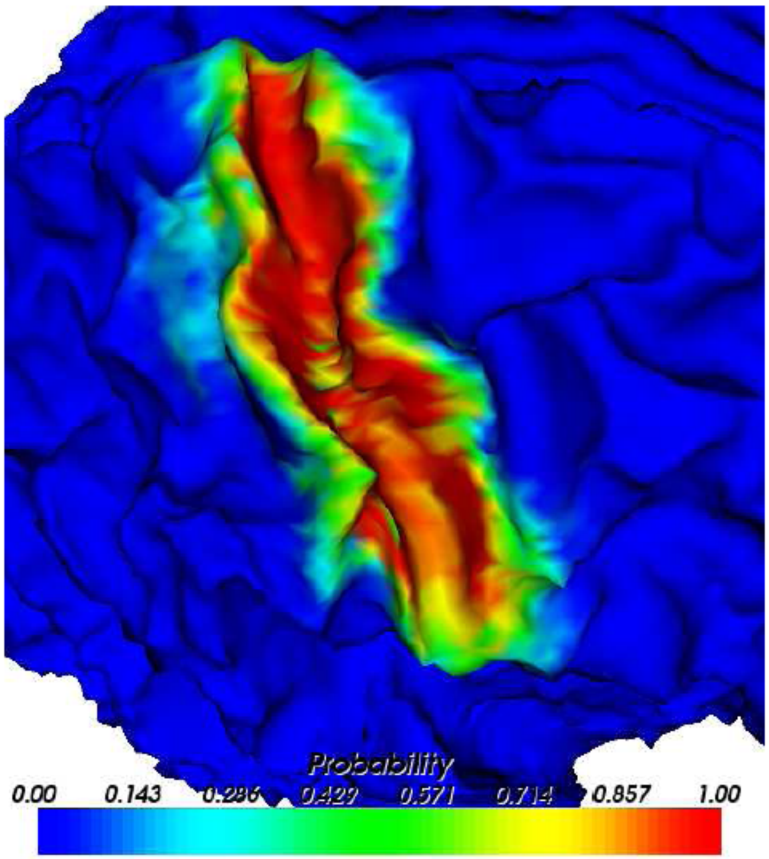
Validation of surface registration based on 149 subjects

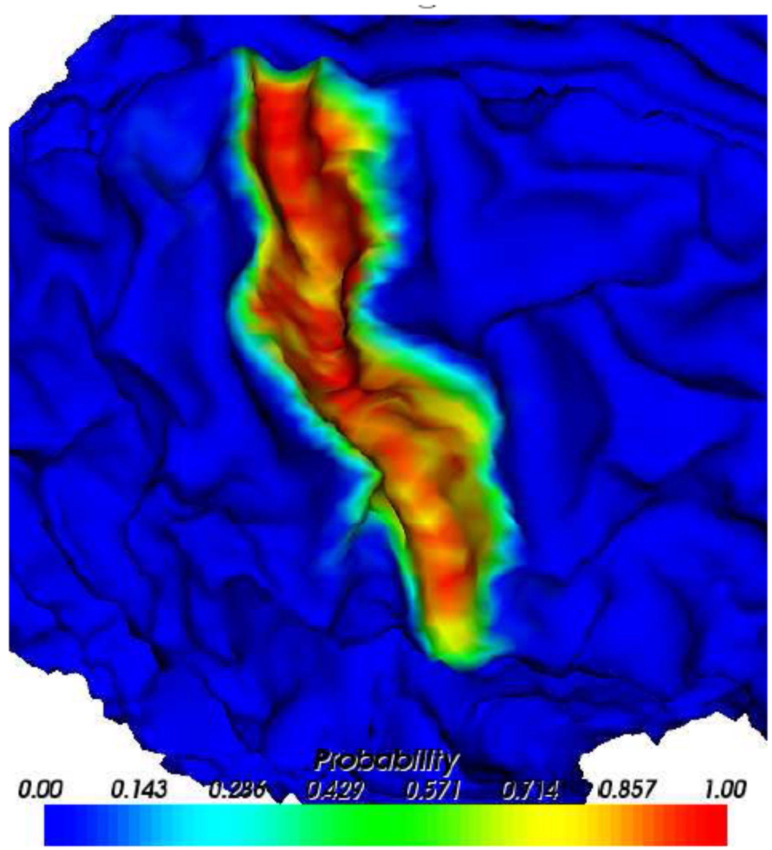


Central and temporal sulci

Traces of sulci provided by Cathia et al. IEEE TMI (2003)

Right central sulcus matching probability





3D volume registration **2D surface registration**

Alternate approach:
spherical harmonic (SPHARM) correspondence

• Surface registration is given implicitly by matching the coefficients of basis functions of deformation.

•Since the deformation is given in terms of smooth basis function, we do not need to worry about increasing the smoothness of deformation.

• This is shown to be optimal in the least squares fashion.

Spherical Harmonic (SPHARM) Representation

- Spherical harmonics are basis functions on a unit sphere.
- SPHARM can be used to construct the Fourier series representation of a functional measurement
- Recent development- Wavelet approach, Weighted-SPAHRM
- SPAHRM has been used in parameterizing anatomical boundary

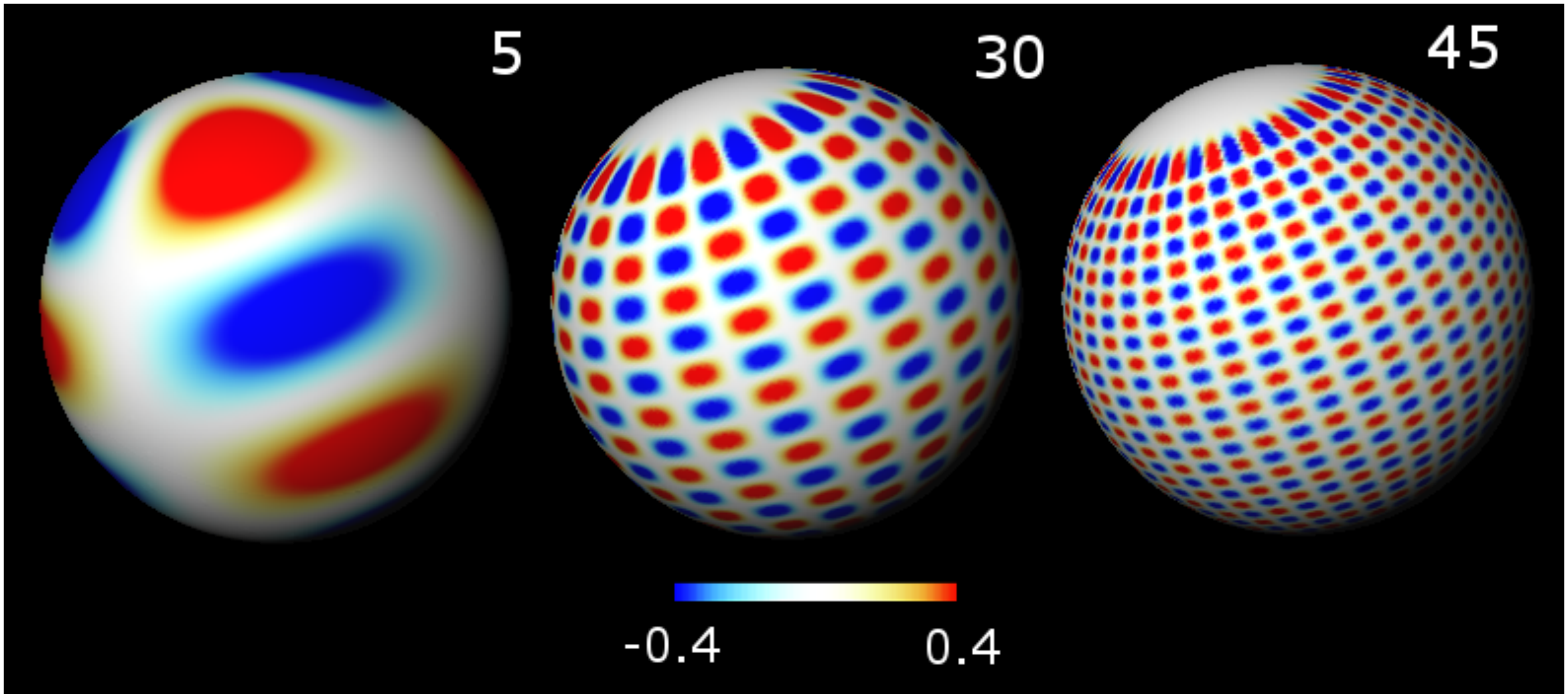
Spherical harmonics

Y_{lm} is called the *spherical harmonic* of degree l and order m .

$$\boxed{Y_{lm} = \begin{cases} c_{lm} P_l^{|m|}(\cos\theta)\sin(|m|\varphi), \\ \frac{c_{lm}}{\sqrt{2}} P_l^0(\cos\theta), \\ c_{lm} P_l^{|m|}(\cos\theta)\cos(|m|\varphi), \end{cases}} \begin{matrix} -l \le m \le -1, \\ m = 0, \\ 1 \le m \le l, \end{matrix}$$

where $c_{lm} = \sqrt{\frac{2l+1}{2\pi}\frac{(l-|m|)!}{(l+|m|)!}}$ and P_l^m is the associated Legendre polynomials of order m.

Spherical harmonic basis of degree 5, 30 and 45



Lower degree ←→ Coarse anatomical detail
Higher degree ←→ Fine anatomical detail

SPHRM representation

•Given functional measurement *f(p)* on a unit sphere, it is modeled as

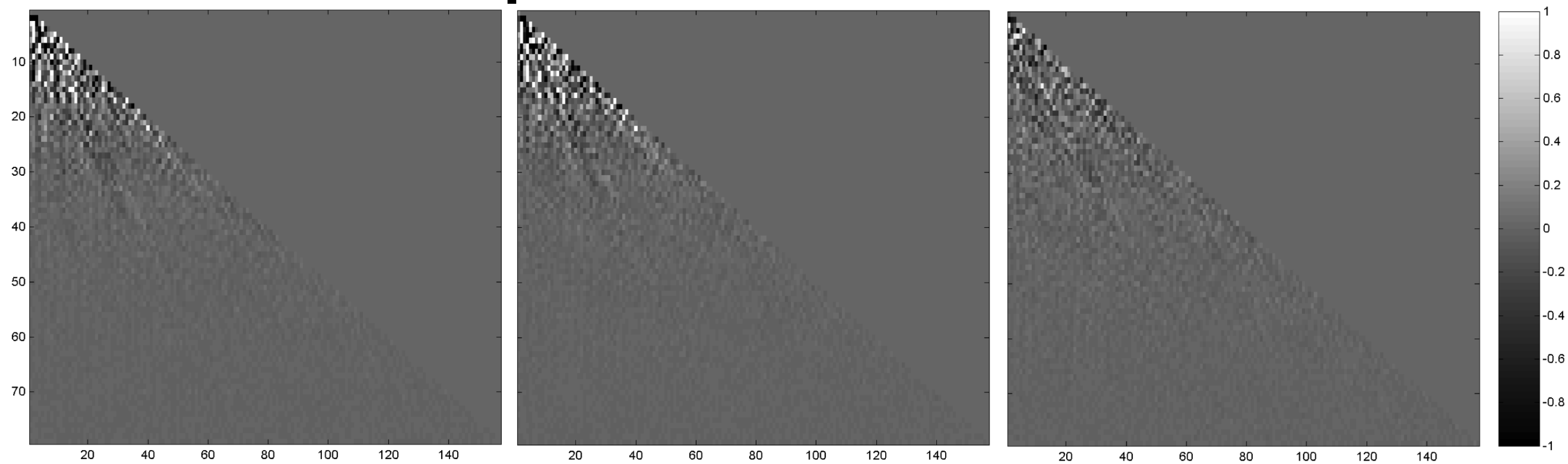
$$f(p) = \sum_{l=0}^{k} \sum_{m=-l}^{l} f_{lm} Y_{lm}(p) + e(p)$$

e: noise (image processing, numerical, biological)

f_{lm}: unknown Fourier coefficients

•The parameters are estimated in the least squares fashion.

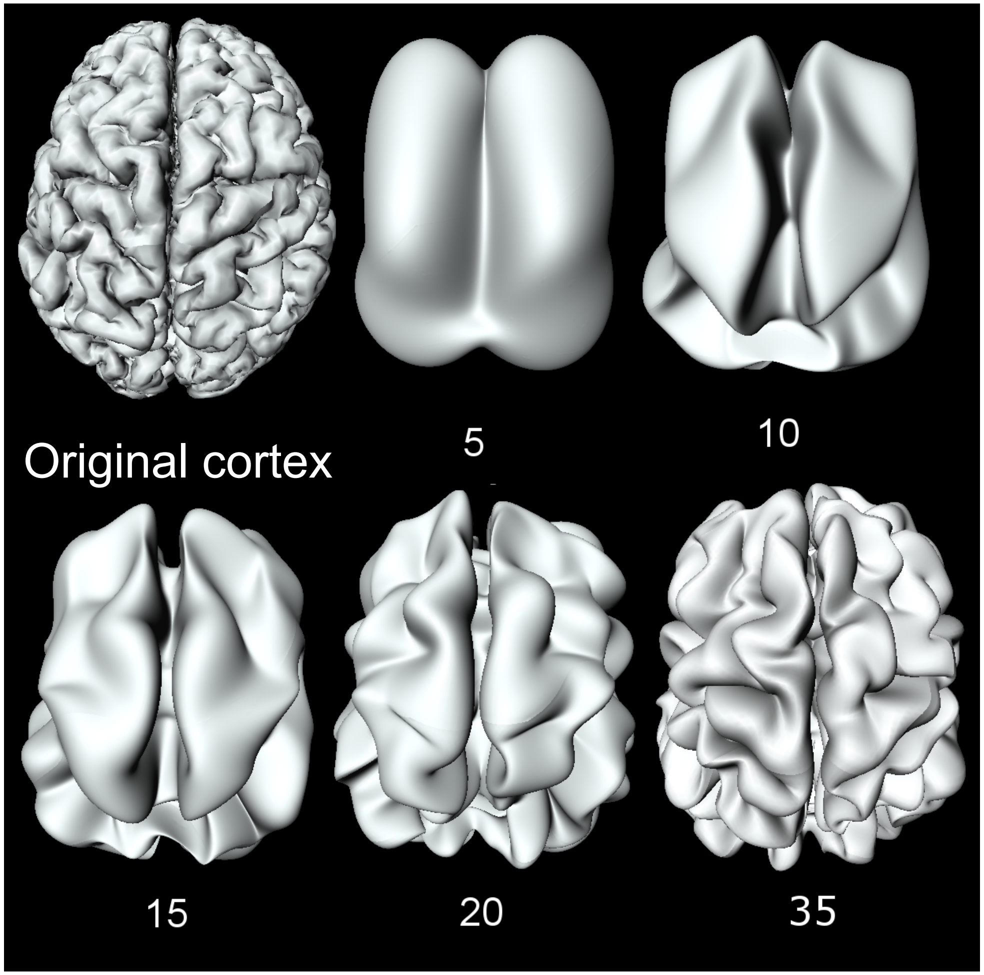
78th degree SPHARM representation



autism **control** **difference**

The coefficients are treated as a multivariate measure and feed into classification techniques.

SPHARM representation up to degree 35



- •Based on direct numerical integration.
- •More than 24 hours of computation.
- •We will show you 10min super fast algorithm

Weighted-SPHARM representation

Initial data: i-th Cartesian coordinate

$$\frac{\partial g}{\partial \sigma} = \Delta g, \; g(p, \sigma = 0) = f(p)$$

Parameter σ controls the amount of smoothing.

The solution is written as the weighted linear combination of spherical harmonics.

Weighted SPHARM is directly related to the following smoothing techniques

1. Diffusion smoothing

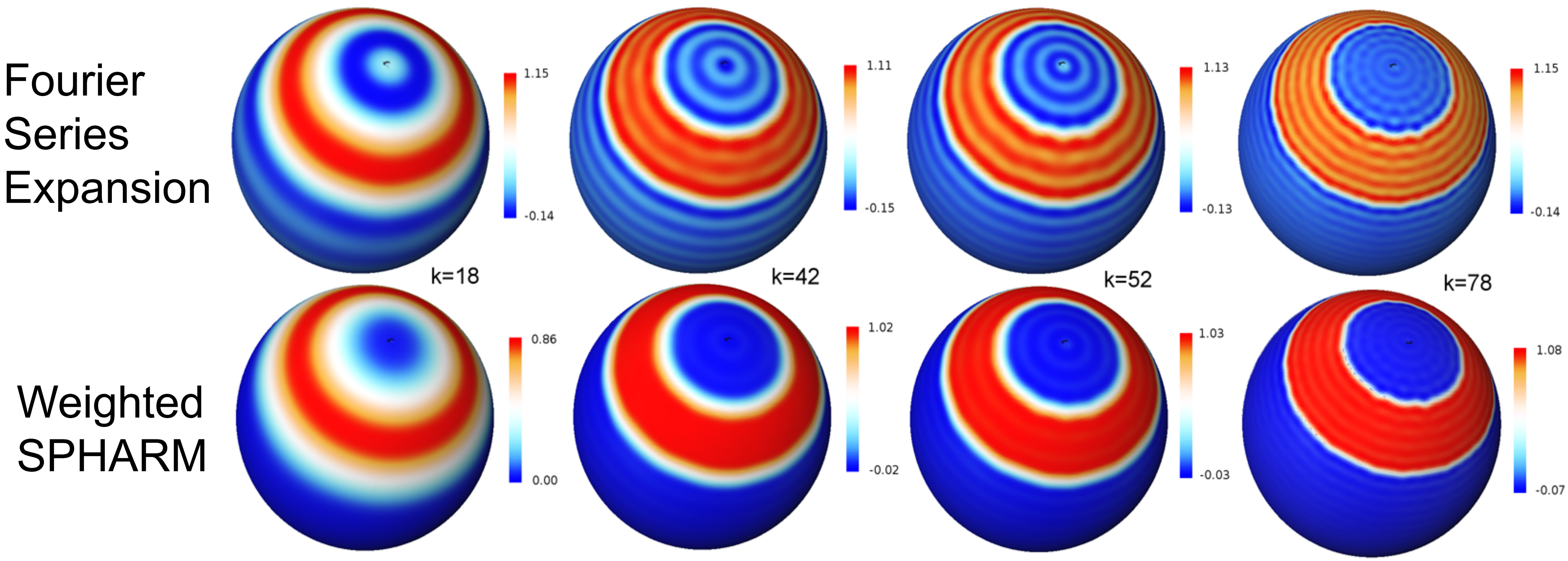
Partial Differential Equation (PDE) based approach with finite element method (FEM).

2. Heat kernel smoothing

Spatially adaptive iterative application of Nadaya-Waton type kernel smoother.

One main advantage of weighted-SPHARM over Fourier series approach:

Reduction of Gibbs effect (ringing artifacts)



$$\sum_{l=0}^{k}\sum_{m=-l}^{l} e^{-l(l+1)\sigma} f_{lm} Y_{lm}$$

Weighted-SPHARM representation

Surface coordinates

$$v(\theta, \varphi) = (v_1(\theta, \varphi), v_2(\theta, \varphi), v_2(\theta, \varphi))$$

Weighted-SPHARM

$$v_i(\theta, \varphi) = \sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} f_{lm}^i Y_{lm}(\theta, \varphi)$$

+ noise

This generalizes the traditional SPHARM representation.

Statistical model on weighted-SPHARM

$$v_i(\theta,\varphi) = \sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} f_{lm}^i Y_{lm}(\theta,\varphi)$$

$$f_{lm}^i \sim N(\mu_{lm}^i, \sigma_l^2)$$

$$\Updownarrow$$

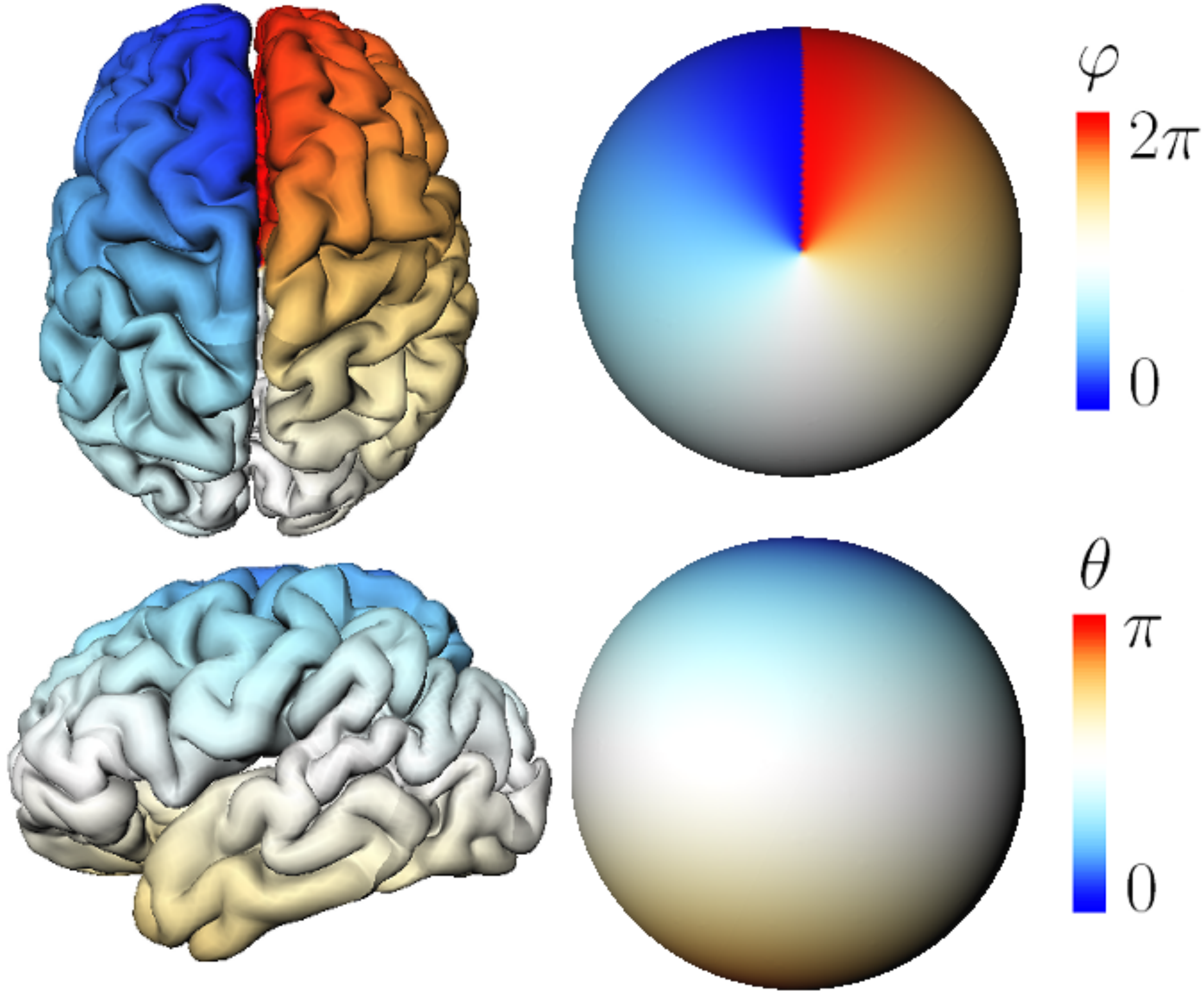
Equivalent to Random Field Theory

$$v_i(\theta,\varphi) = \sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} \mu_{lm}^i Y_{lm}(\theta,\varphi) + \epsilon(\theta,\varphi)$$

Coordinates on cortex

Need for a spherical mapping with the least metric distortion:

Quasi area preserving algorithm



$T_1, T_2, \cdots, T_n$:*n* triangles in the cortex

$f(T_1), f(T_2), \cdots, f(T_n)$:deformed triangles

$\|T_i\|$ area of a triangle.

Flatten the cortex in such a way that

$$\frac{\|f(T_1)\|}{\|T_1\|} = \frac{\|f(T_2)\|}{\|T_2\|} = \cdots = \frac{\|f(T_n)\|}{\|T_n\|} = \lambda$$

for some constant . There are *n* equations but approximately 3/2*n* deformation parameters to estimate so there will be infinite number of area-preserving flat maps. Among all solutions, we find one solution that minimize the sum of squared errors:

Quasi area preserving algorithm

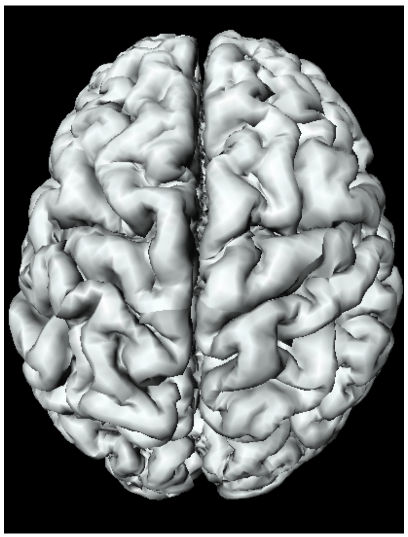
$$L(f) = \sum_{i=1}^{n} \left(||T_i|| - \lambda ||f(T_i)|| \right)^2$$

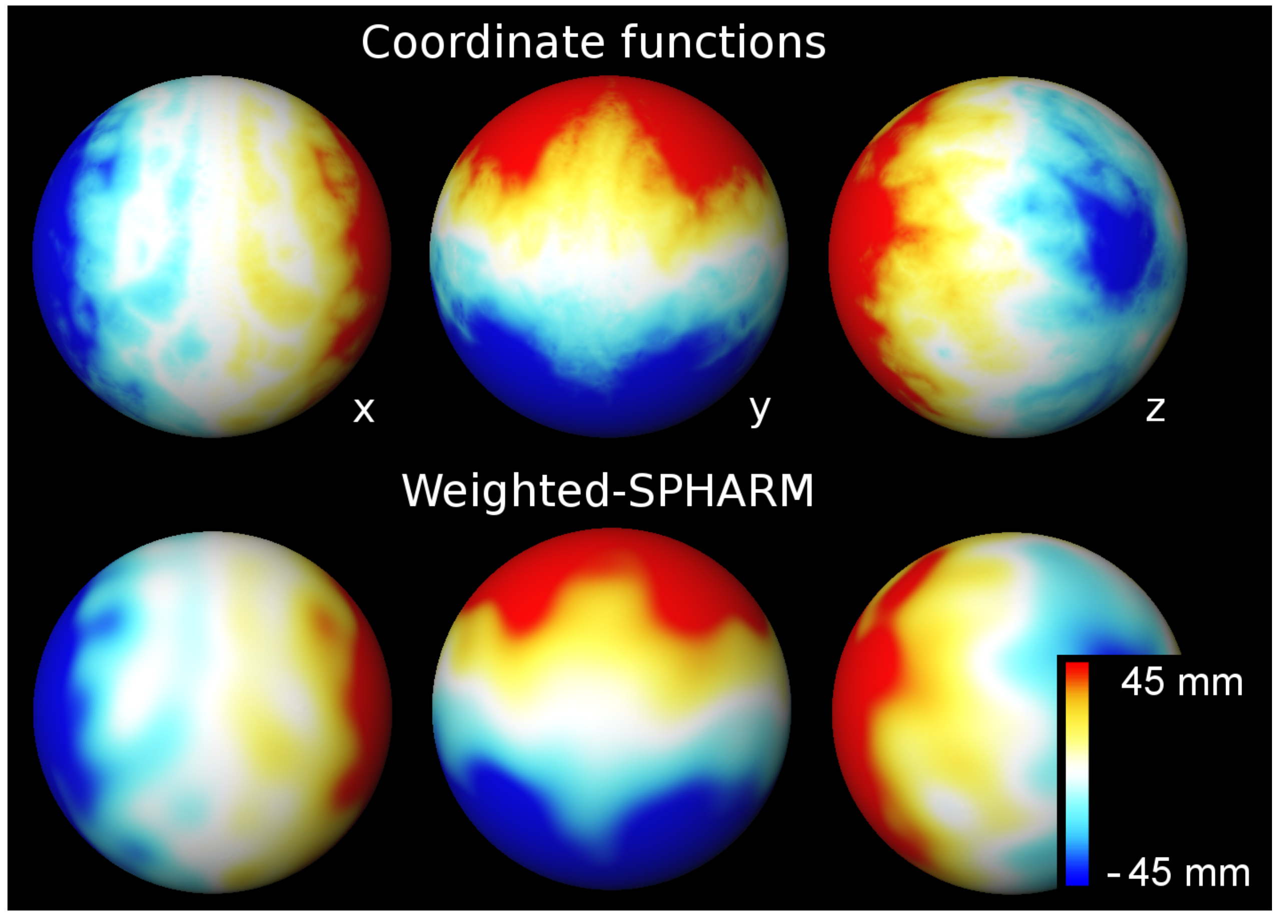
- Gradient descent method is used to find the minimum. In each iteration step, the position of every vertices are slightly moved to locally minimize *L* until it is not possible to minimize any further.

One possible research project. A student was working on this problem but did not finish it.

Mapping from cortex to unit sphere

Each x, y, z Cartesian coordinates are represented independently.





How do we estimate SPHARM coefficients numerically?

Weighted-SPHARM

$$v_i(\theta,\varphi) = \sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-l(l+1)\sigma} f_{lm}^{i} Y_{lm}(\theta,\varphi)$$

15,000 Fourier coefficients

Iterative residual fitting (IRF) algorithm.

See Chung et al. TMI (2007) for detail.

It will not be discussed here.

Previous approach in estimating Fourier coefficients

- For each point p_i , we have measurement $f(p_i)$.
- Corresponding Fourier series:

$$f(p_i) = \beta_0 \phi_0(p_i) + \beta_0 \phi_0(p_i) + \cdots + \beta_k \phi_k(p_i)$$

- Matrix form:

$$F = \Phi \beta$$

$$\beta = (\Phi^{'} \Phi)^{-1} \Phi^{'} F$$

- This is a nontrivial linear problem

Estimating 15,000 Fourier coefficients

Direct numerical integration takes forever.

Fast Fourier transform (FFT) is not fast either.

Iterative residual fitting (IRF) algorithm

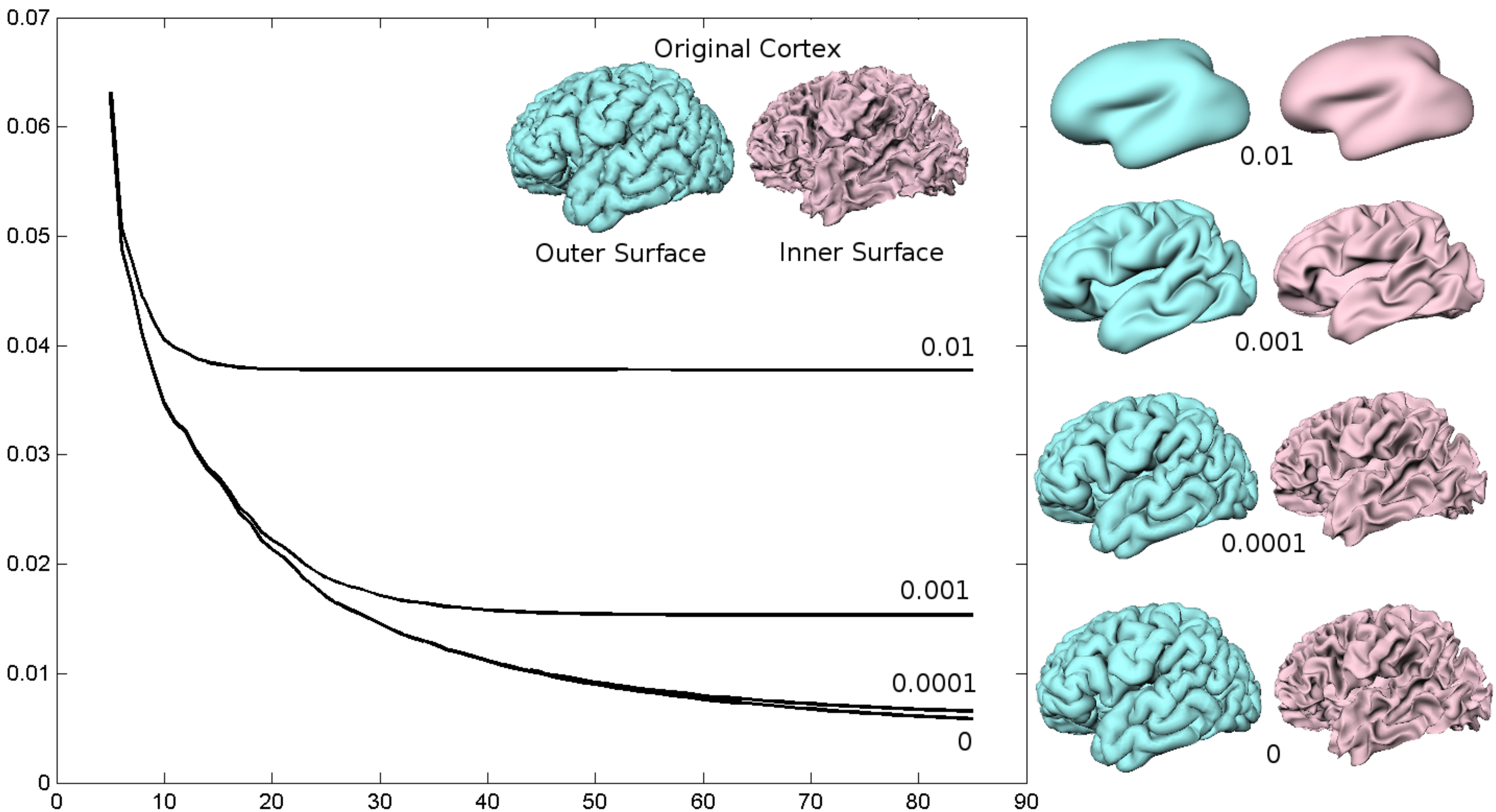
1. Estimate the Fourier coefficients iteratively from lower degree to higher degree.
2. Break one huge linear problem (3GB) into many smaller linear problems (500MB).
3. At each iteration, residual is used to estimate the coefficients of next degree.

Iterative residual fitting (IRF) algorithm

MATLAB implementation can be downloaded from
http://www.stat.wisc.edu/~mchung/softwares/weighted-SPHARM/weighted-SPHARM.html

Sample cortical surface data is also provided.

Weighted-SPHARM at the 80th degree for different bandwidth



Root mean squared error (RMSE)
= error between original surface and weighted-SPHARM

Determining the optimal degree via stepwise forward model selection framework

Consider the following $(k-1)$-th degree model

$$f(p_i) = \sum_{l=0}^{k-1} \sum_{m=-l}^{l} e^{-\lambda(l+1)\sigma} f_{lm} Y_{lm}(p_i) + \epsilon(p_i),\ i = 1, \cdots, n$$

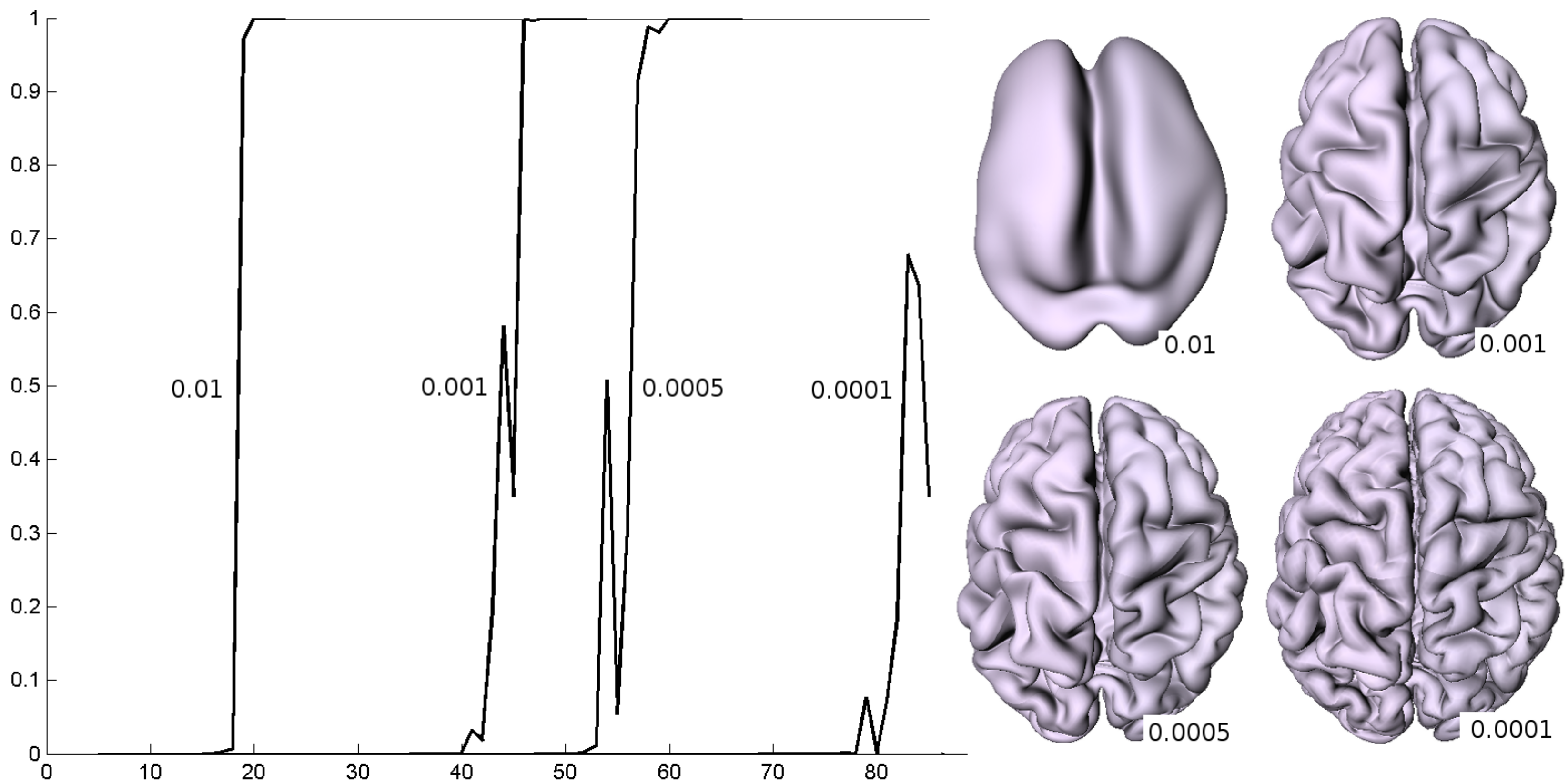
where ϵ are Gaussian random variables. Testing if the k-th degree model is better than the previous $(k-1)$-th degree model can be done by testing

$$H_0 : f_{km} = 0 \text{ for all } -k \le m \le k.$$

Then under the null hypothesis, the test statistic is

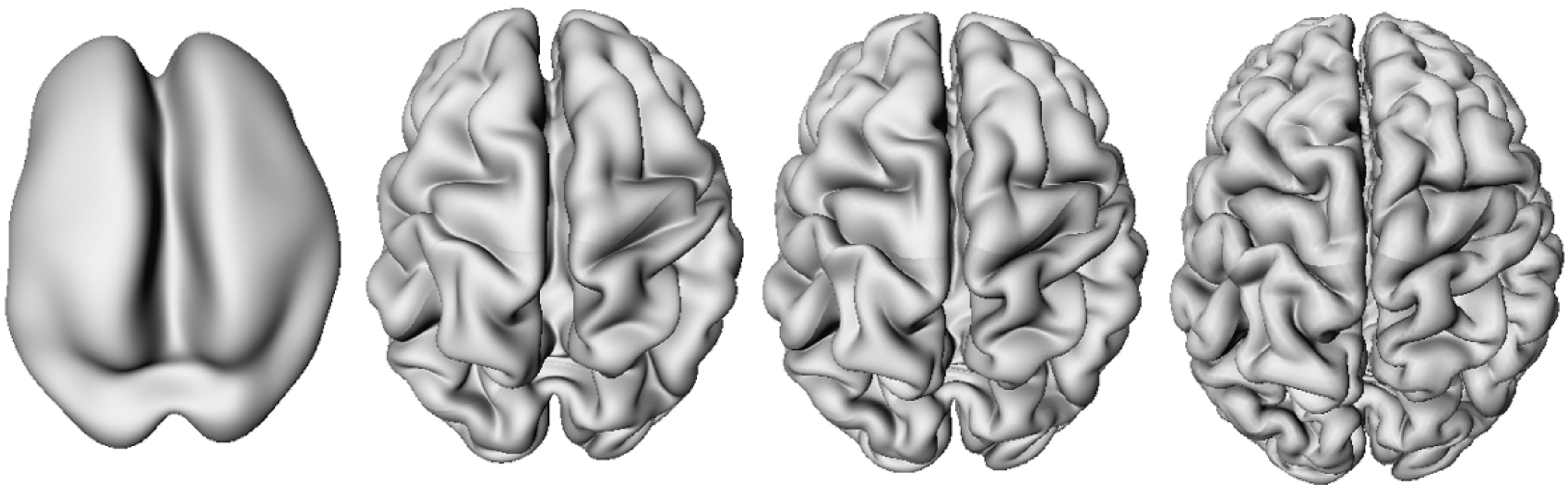
$$F = \frac{(\text{SSE}_{k-1} - \text{SSE}_k)/(2k+1)}{\text{SSE}_{k-1}/(n-(k+1)^2)} \sim F_{2k+1, n-(k+1)^2}$$

For each bandwidth σ , optimal degree is automatically selected via **forward best model selection procedure**.



Optimal degree= first P-value >0.05

Weighted-SPHARM at different bandwidth

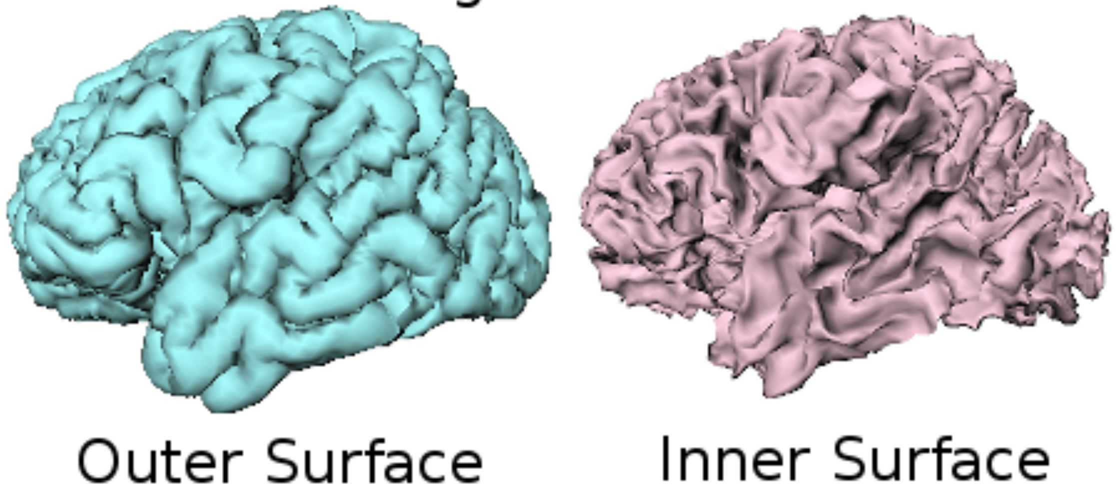


0.01 0.001 0.0005 0.0001

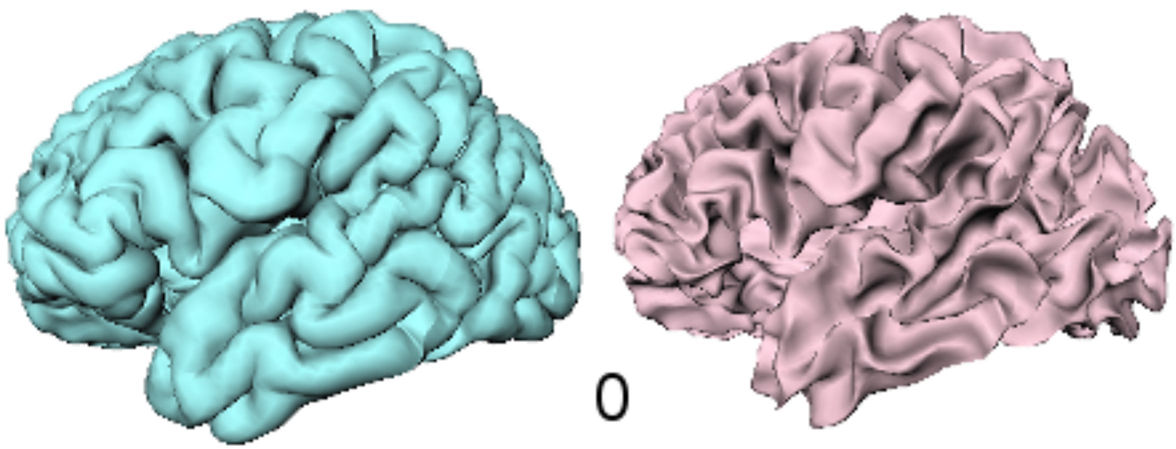
- The degree is selected automatically.

- The only free parameter in the model is the bandwidth.

Original Cortex



80 degree SPHARM

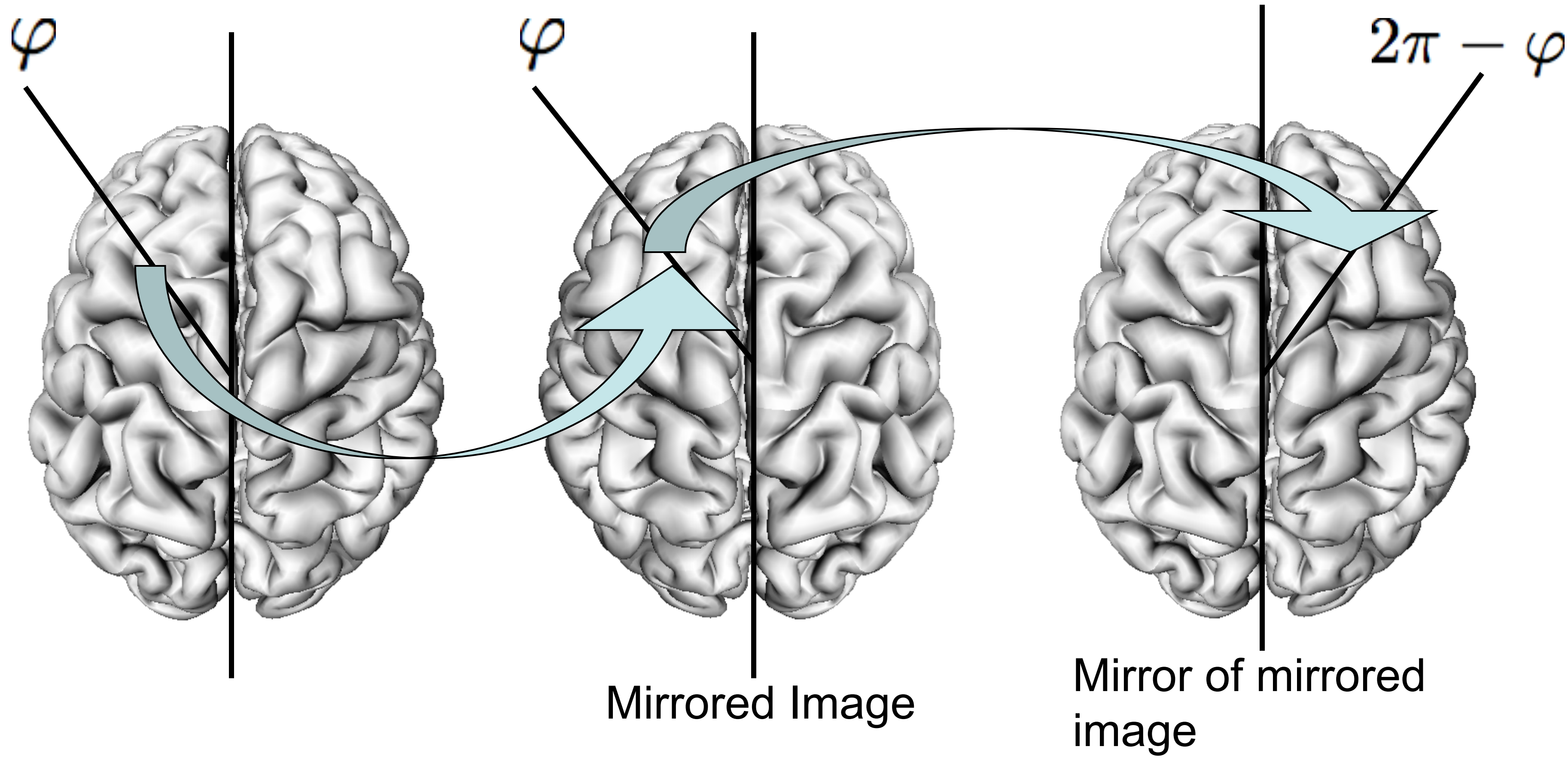


Surface registration via SPHARM-correspondence

- The correspondence minimizes the sum of squared errors.

- Simply match the SPHARM coefficients. Most classification techniques are based on this idea.

Establishing Hemispheric correspondence



$$Y_{lm}(\theta, 2\pi - \varphi) = \left\{ \begin{array}{cc} -Y_{lm}(\theta, \varphi), & -l \leq m \leq -1, \\ Y_{lm}(\theta, \varphi), & 0 \leq m \leq l. \end{array} \right\}$$

Consistent definition using weighted-SPHARM

Outer surface v_i

Fourier coefficients f^i_{lm}

Inner surface w_i

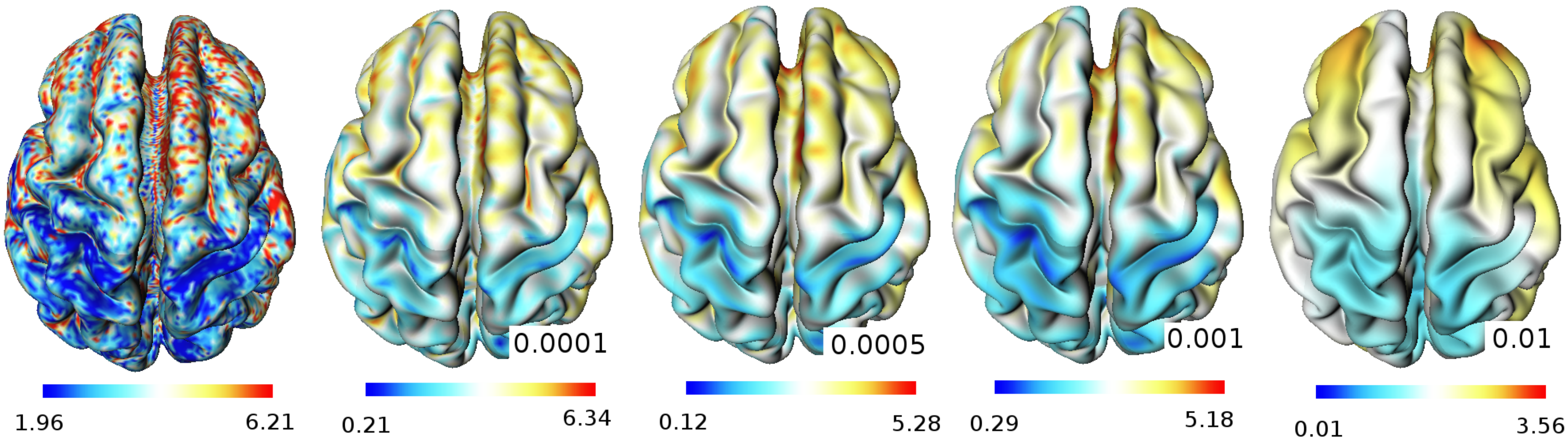
Fourier coefficients g^i_{lm}

Cortical thickness

$$\sum_{l=0}^{k} \sum_{m=-l}^{l} e^{-\lambda(\lambda+1)\sigma} [\sum_{i=1}^{3} (g^i_{lm} - f^i_{lm})^2]^{1/2}$$

MIAR (2006), TMI (2007)

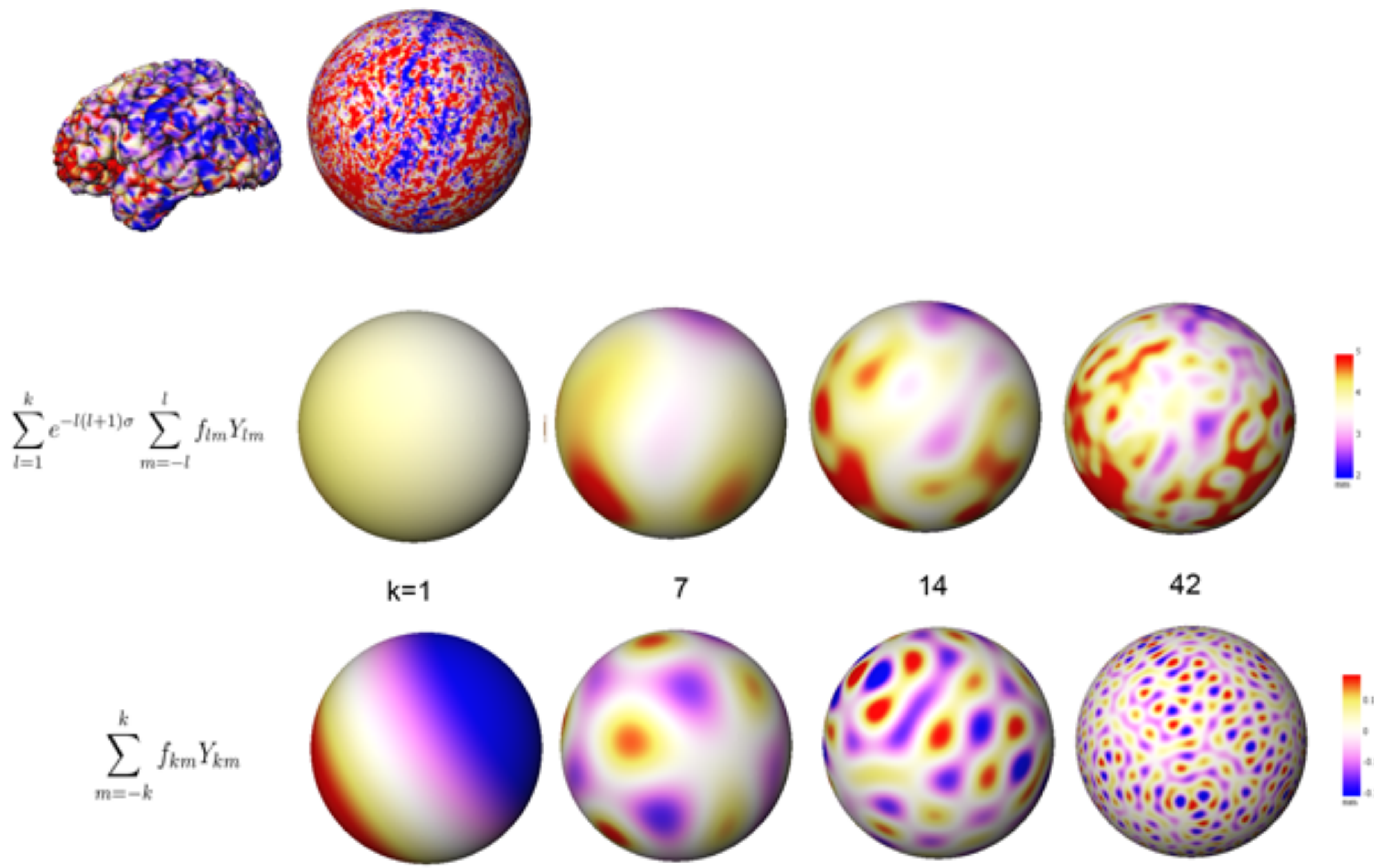
SPHARM estimation of cortical thickness



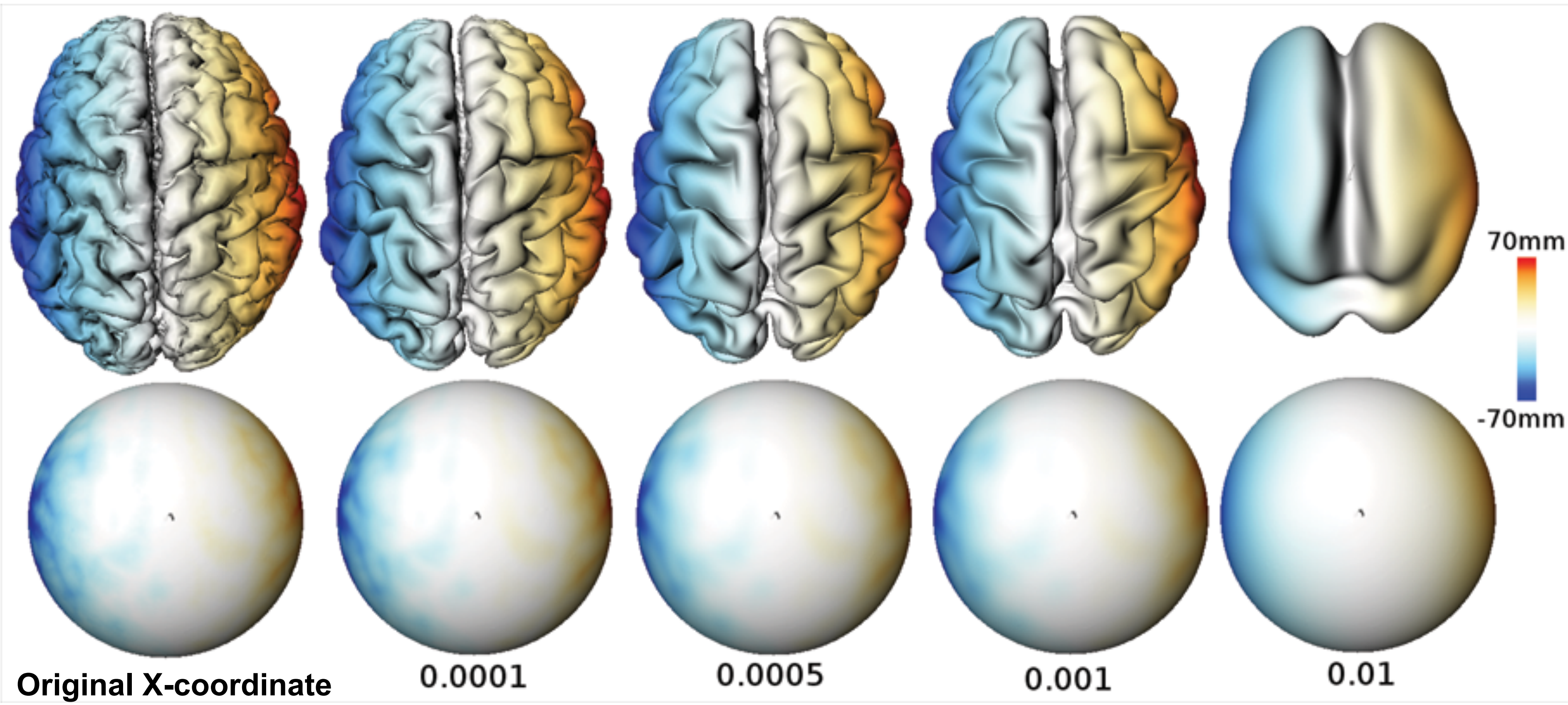
Thickness estimation based on traditional method

Too much smoothing

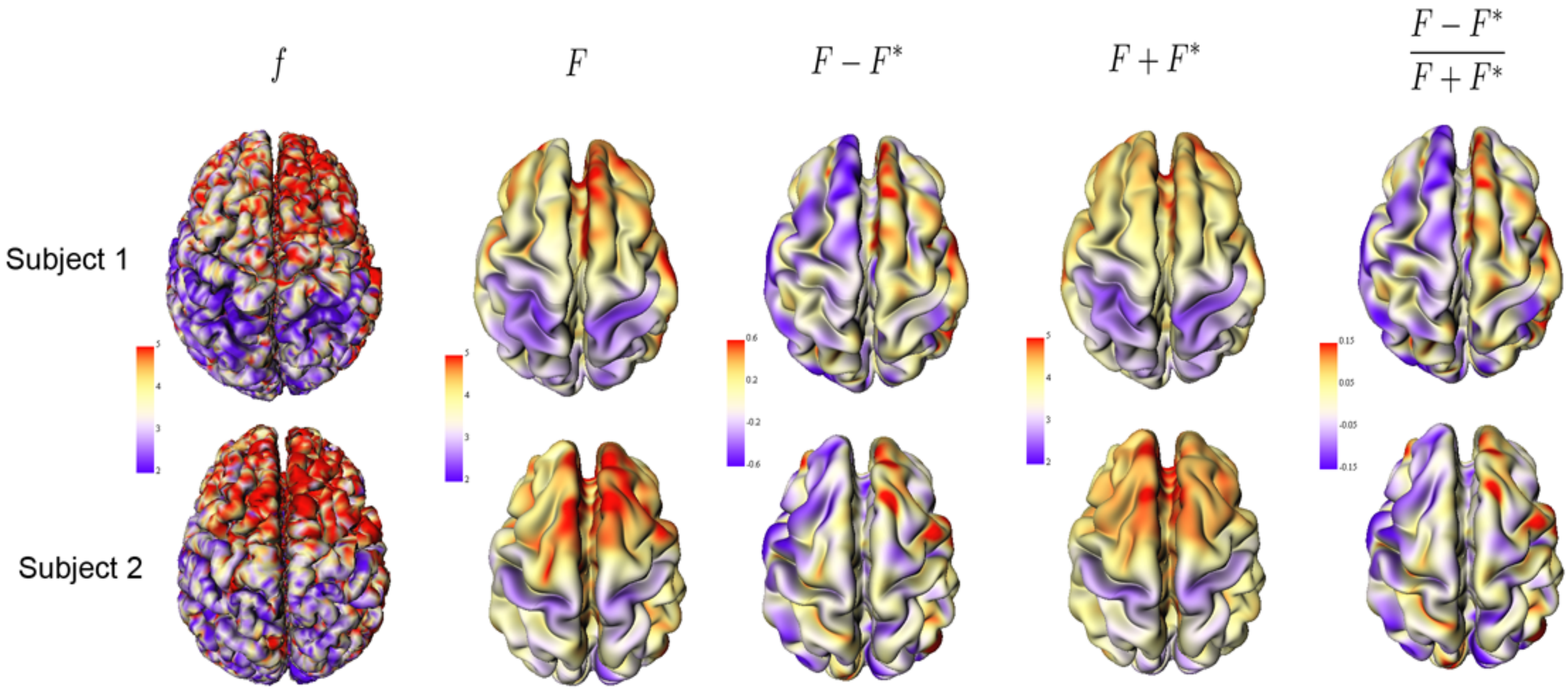
Weighted-SPHARM of cortical thickness



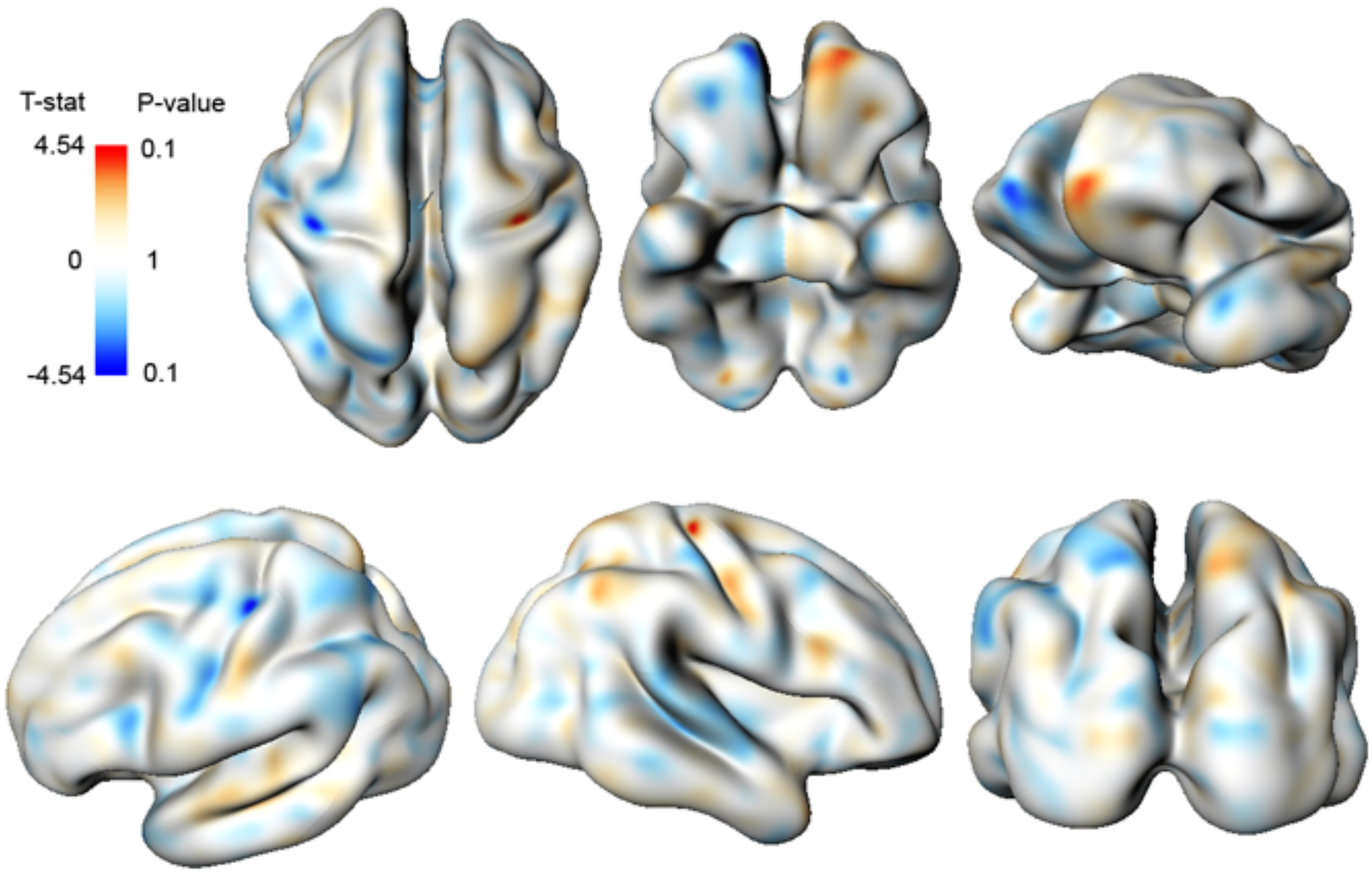
Weighted-SPHARM at different scale



Cortical Asymmetry Index



Cortical Asymmetry Result



Application: correlating anatomy with nonimaging measures

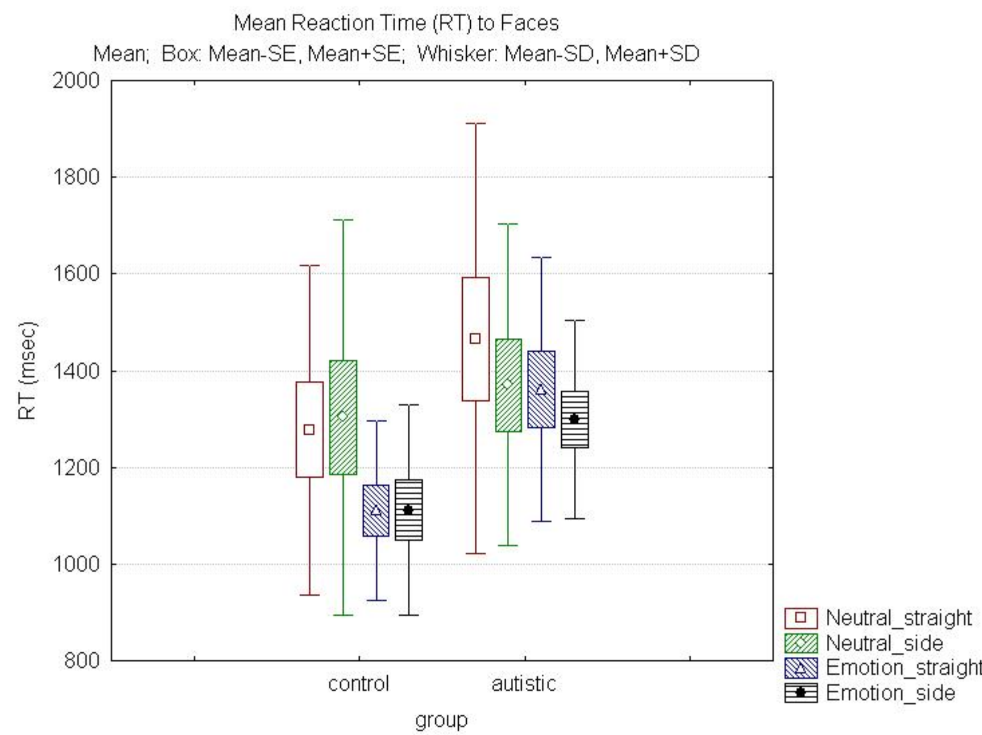
Facial emotion discrimination task response time

2 (Emotion) × 2 (Orientation)

Neutral Emotional

Straight-ahead

Quarter-turned



Dalton et al. (2005)

24 emotional faces, 16 neutral faces

Let $Y = (Y_1, Y_2)$ be two variables of interests and $X = (X_1, \cdots, X_p)$ be a row vector of multivariate that should be removed. For instance, we may let Y_1 to be the cortical thickness and Y_2 be the face recognition tasks while X_1 and X_2 are age and total surface area. The covariance matrix of $(Y, X)'$ is given by

$$\boxed{\mathbb{V}(Y,X)' = \begin{pmatrix} \Sigma_{YY} & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_{XX} \end{pmatrix}} \tag{1}$$

Note Σ_{YY}, Σ_{YX} and Σ_{XX} are the cross covariance matrices. Then the partial covariance of Y given X is

$$\boxed{\Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY} = (\sigma_{ij})}.$$

The *partial correlation* $\rho_{Y_i,Y_j|X}$ is then defined as the correlation between variables Y_i and Y_j while controlling for other variables X and given by

$$\boxed{\rho_{Y_i,Y_j|X} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}.}$$

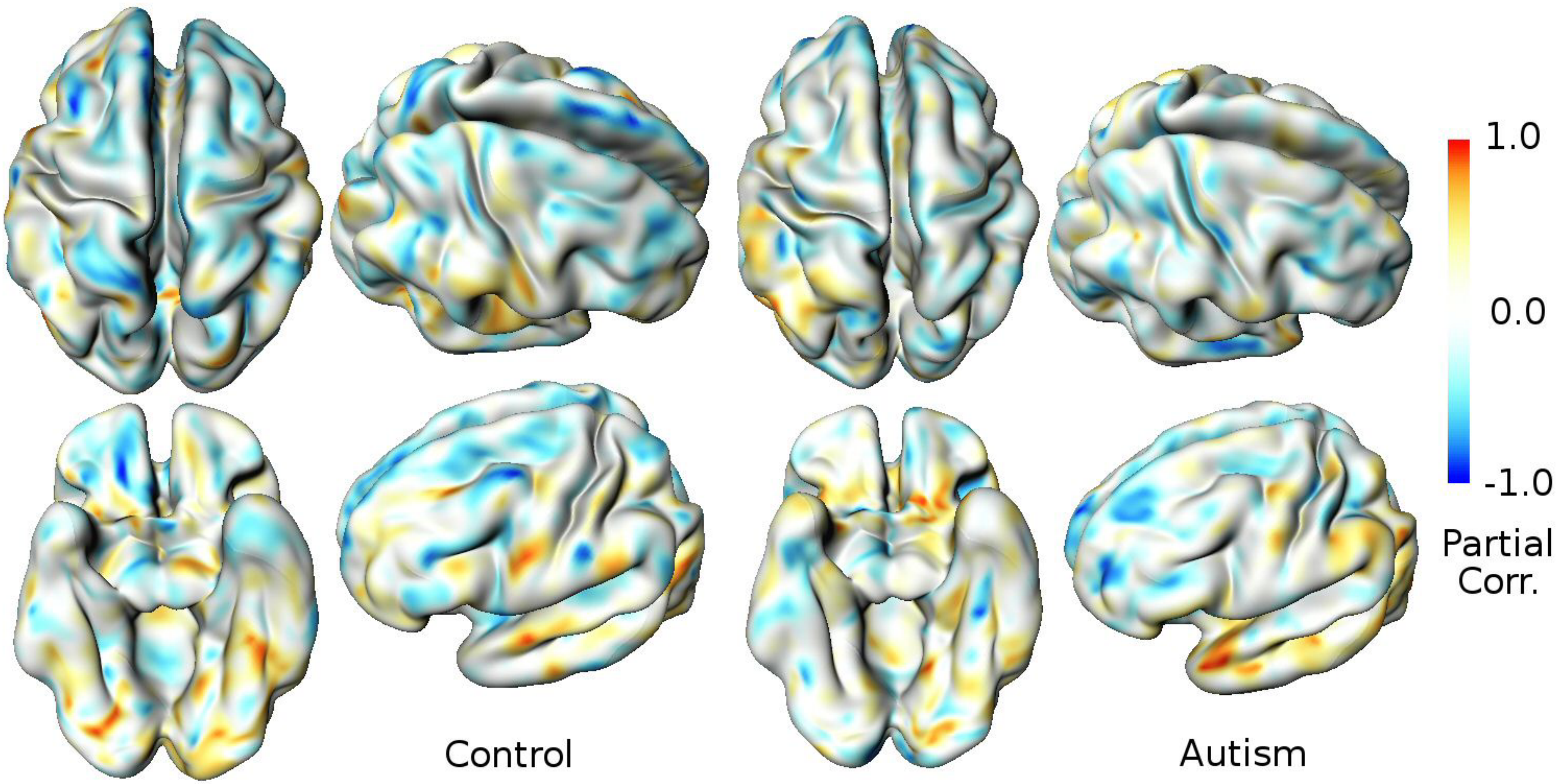
The reason we are using the *conditional* notation $|$ in defining the partial correlation is due to the fact that the partial correlation is equivalent to *conditional correlation* if $\mathbb{E}(Y|X) = a + BX$ for some vector a and matrix B, which is true under normal assumption. This is the formulation we used to compute the par-

See

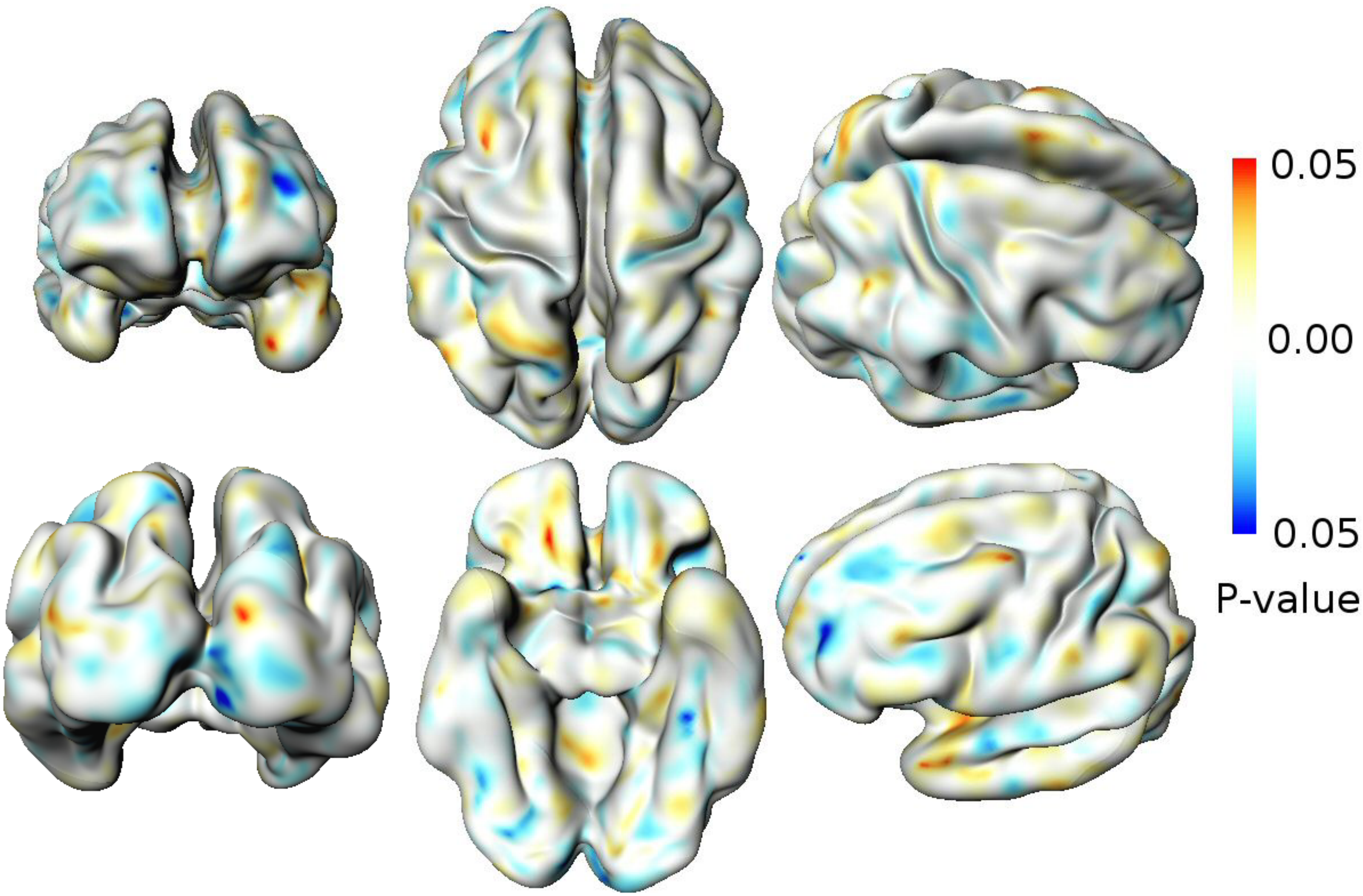
http://www.stat.wisc.edu/~mchung/papers/partial.correlation.TR1109.2005.pdf

detailed explanation on partial correlation

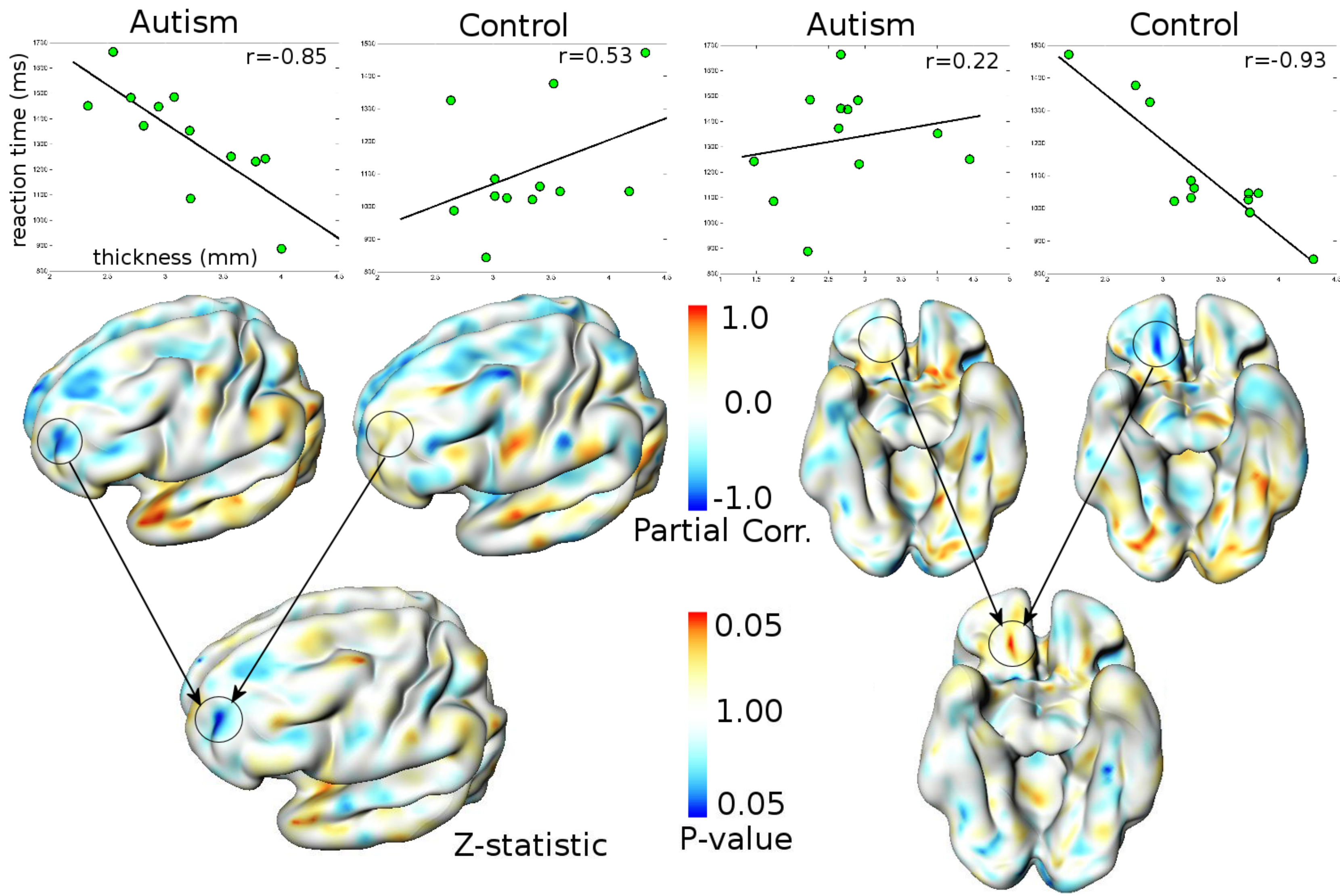
Correlation between response time and cortical thickness
(the effect of age and brain size difference has been removed)



The final SPM showing statistically significant correlation difference between the groups.

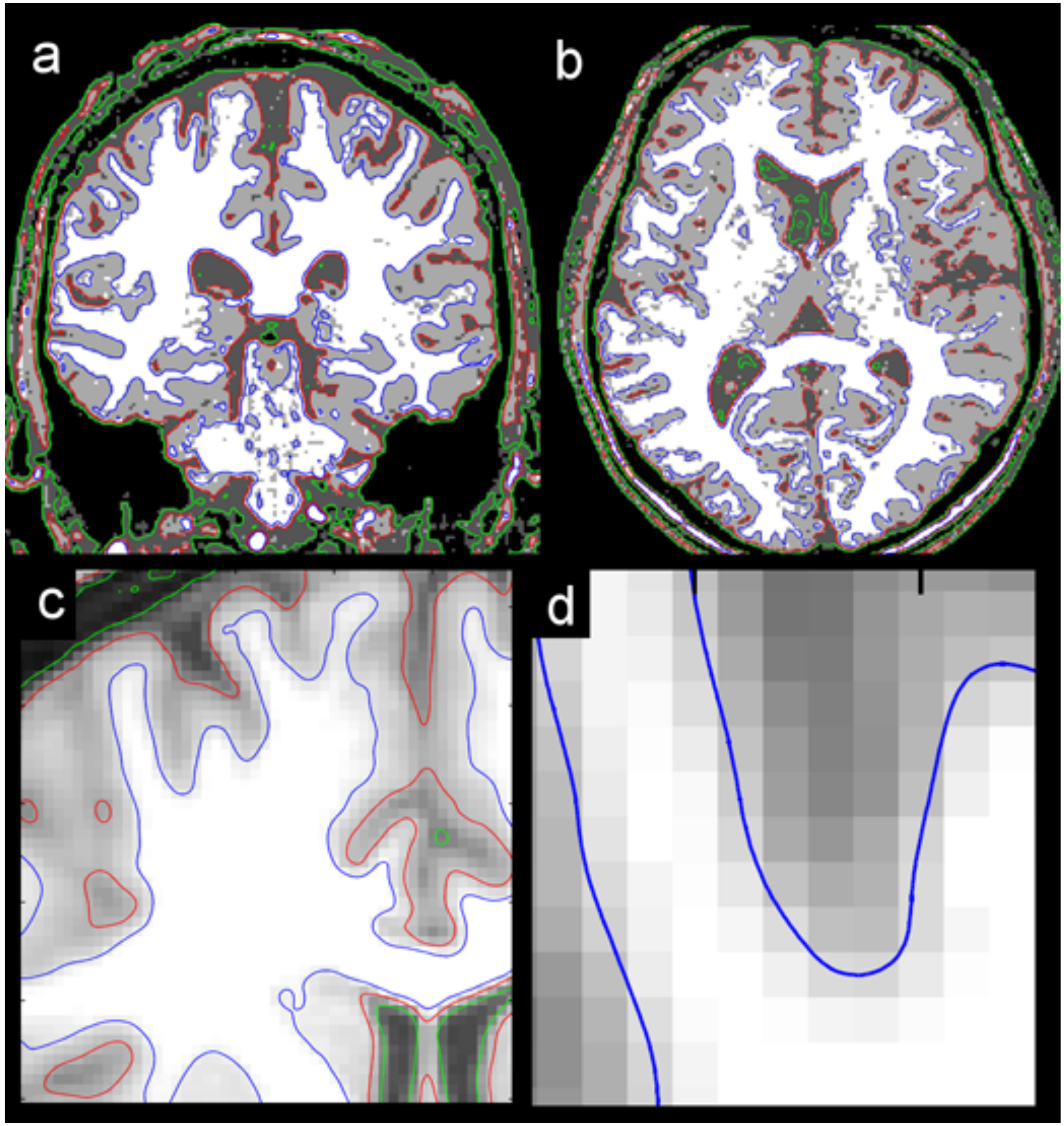


Scatter plot of response time vs. thickness



Thin Plate Spline (TPS) segmentation and modeling

TPS represents anatomical boundary as the zero level set of smooth function consists of polynomial and radial basis functions (Wahba, 1990; Xie et al., 2005a).



Thin Plate Spline

Measurement *f* is represented as

$$f(p) = \sum_i \alpha_i \phi_i(p) + \sum_j \beta_j \varphi(p - p_j)$$

where ϕ_i is polynomial basis and φ is the TPS radial basis

Parameters are estimated by minimizing

$$\min_f \sum_{i=1} \left| y_i - f(p_i) \right|^2 + \lambda J_3^2(f),$$